Math 143

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1 September 27

1.1 Parameterized Equations and Curves

Def.

- **A parameterized, smooth curve** (i.e. continuously differentiable up to a specified order) in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is a function $\vec{\alpha}((t))$: $\mathbb{R}^1 \rightarrow \mathbb{R}^2$.

  Ex. $\vec{\alpha}(t) = \begin{bmatrix} t \\ 1-|t| \end{bmatrix}$ is not differentiable at $t = 0$, and is therefore not a parameterized curve. This point of non-continuous differentiability is called a singularity.

  Ex. $\vec{\alpha}(t) = \begin{bmatrix} sin(\frac{2t}{3}) \\ cos(t) \end{bmatrix}$, $t \in [0, 6\pi]$, self-intersects itself at $t = 0$ with the tangent lines at differing times having different directions. While $\vec{\alpha}((t))$is continuously differentiable, we need to specify the time t_0 at which each of the tangent vectors at the intersection differ.

- The **Tangent Vector** is how this curve changes: $\vec{\alpha}'(t) = \lim_{n\to\infty} \frac{\vec{\alpha}(t+h)-\vec{\alpha}(t)}{h}$.

  Example: Let $\vec{\alpha}((t))= \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \end{bmatrix}$, then $\vec{\alpha}'(t) = \begin{bmatrix} \alpha_1'(t) \\ \alpha_2'(t) \\ \alpha_3'(t) \end{bmatrix}$.

- **Tangent Line** of $\vec{\alpha}((t))$at $t = t_0$ is $y(s) = \vec{\alpha}(t_0) + s\vec{\alpha}'(t_0)$.

- A **Singularity** is a point on a parameterized curve when at least one partial derivative is zero.

- The **Speed** of a parameterized curve $\vec{\alpha}((t))$is $s(t) := |\vec{\alpha}'(t)| = \sqrt{\alpha_1'(t)^2 + \alpha_2'(t)^2 + ...}$

- A curve is **regular** if $\vec{\alpha}'(t) \neq \vec{0}$. Equivalently, $s(t) \neq 0$.

  A reparameterization of $\vec{\alpha}((t))$is a curve $\vec{B}(s) := \vec{\alpha}(t(s))$ where t(s) is some increasing or decreasing function. otherwise it might not be unique? – two values map to the same point? We say this function doesn't need to be monotonic?

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \\ S \arrow[u, "t(s)"] \arrow[ur, "\vec{B}=\vec{\alpha}\circ t"'] & \end{tikzcd}$$

Ex.

- $\vec{\alpha}((t))= \begin{bmatrix} t^3 \\ t^6 \end{bmatrix}$ is not regular since at $t = 0$, $\vec{\alpha}'(t) = \vec{0}$

  Remark: We can reparameterize $\vec{\alpha}((t))$to a curve $\vec{B}(s)$ using the parameterization $s = t^3$. Then $\vec{B}(s) = \begin{bmatrix} s \\ s^2 \end{bmatrix}$ is regular since $\vec{B}'(s) \neq 0\ \forall s \in$ domain.

- $\vec{\alpha}((t))= \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$ is regular.

1.2 Curve Lengths

Def.

- The **Length** of $\vec{\alpha}((t))$between t_1 and $t_2 \in \mathbb{R}^1$ is L $= | \int_{t_1}^{t_2} |\vec{\alpha}'(t)|\, dt\ |$.
- The **signed arclength** of a curve $\vec{\alpha}((t))$from $t_0 \to t_1$ is $S(t) = \int_{t_0}^{t_1} |\vec{\alpha}'(u)|\, du$.
- We say $\vec{\alpha}((t))$is **unit speed** if $|\vec{\alpha}'(t)| = \vec{1}$ for all $t \in$ domain.

  Remark: When $\vec{\alpha}((t))$is of unit speed, the length of $\vec{\alpha}((t))$from $t_1 \to t_2$ is L $= t_2 - t_1$.

- An curve is **arclength parameterized** if $v(t) = |\vec{\alpha}'(t)| = 1$.

Theorem 1.1 *Any regular curve has an arc-length parameterization. Equivalently, for any regular curve $\mathbb{R}^1 \to \mathbb{R}^3$, $\exists$ t: $\mathbb{R}^1 \to \mathbb{R}^1$ that is increasing such that $\vec{B} = \vec{\alpha} \circ t$ has unit speed.*

Proof. todo.

2 September 29

2.1 Frenet Frames

Def.[Frame] A frame along a curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ (also known as a moving frame of α) is a family of orthonormal basis $(e_1(t), e_2(t), e_3(t))$ such so that $e_3 = e_1 x e_2$.

- This is a "frame" or "moving frame" along a curve $\alpha(t)$.
- $\vec{e}_2 \perp \vec{e}_1$ along $\alpha(t)$.
- $\vec{e}_3$ can be found with the cross product.

Lemma 2.1 *For a frame $(e_1(t), e_2(t), e_3(t))$, we have:*

1. $e_i'(t) \cdot e_j(t) = -e_i(t) \cdot e_j'(t)$
2. $e_i'(t) \cdot e_i(t) = 0$ *(can be derived from 1.).*

Proof: $e_i(t) \cdot e_j(t) = \delta_{ij}$ [Property of orthonormal basis].

$$\delta_{ij} = \left\{ \begin{array}{cc} 1, & \text{if i = j} \\ 0, & \text{otherwise} \end{array} \right\}$$

Which naturally follows from $e_i(t)$ and $e_j(t)$ forming an orthonormal basis of $\alpha(t)$. Then $\frac{d}{dt}[e_i(t) \cdot e_j(t)] = e_i'(t) \cdot e_j(t) + e_i(t) \cdot e_j'(t) = \frac{d}{dt}[\delta_{ij}] = 0$.

Lemma 2.2 *$(e_1', e_2', e_3') = [e_1, e_2, e_3] \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix}$. We define $\begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix} = W$ and see that $W^T = -W$.*

Proof: Use the last lemma [left as an exercise].

2.2 Curvature and Torsion

Note. Together, curvature and torsion determine the shape of a curve.
Def. Suppose $\alpha : \mathbb{R} \to \mathbb{R}^3$ is a regular curve **parameterized by arclength**. Let T(t) = $\alpha'(t)$ be the unit tangent vector. We say $k(t) = |\alpha''(t)| = |T'(t)|$ is the "curvature". In other words, the norm of the second derivative of α is the curvature. But note, it **must be parameterized by arclength** (for this definition to apply).

1. If $k(t) = 0$, for all $t \in \mathbb{R}$, then $\alpha''(t) = 0$. $\Rightarrow T(t) = \alpha'(t) =$ constant.
2. Unit circle in $\mathbb{R}^3$: $\alpha(t) = \begin{bmatrix} cos(t) \\ sin(t) \\ 0 \end{bmatrix}$, $\alpha''(t) = \begin{bmatrix} -cos(t) \\ -sin(t) \\ 0 \end{bmatrix}$. Then $\alpha(t) = -\alpha(t) \Rightarrow k(t) = |\alpha''(t)| = |-\alpha(t)| = 1$.
3. Generalized Circle: $\alpha(t) = r \begin{bmatrix} cos(t) \\ sin(t) \\ 0 \end{bmatrix} \Rightarrow \alpha'(t) = \begin{bmatrix} \frac{t}{r}cos(t) \\ \frac{t}{r}sin(t) \\ 0 \end{bmatrix} \Rightarrow \alpha''(t) = \frac{-1}{r^2}\alpha(t)$. Note $\alpha'(t)$ is in arclength parameterization. Therefore, k(t) = $|\alpha''(t)| = \frac{1}{r^2}|\alpha(t)| = \frac{1}{r^2} * r = \frac{1}{r}$ and the curvature is a constant $\frac{1}{r}$ such that the larger the circle (i.e. the larger the radius r), the smaller the curvature.

Def. A curve that is biregular is both regular and $k \neq 0$ (i.e. non-zero curvature).

Theorem 2.3 *Let $\alpha(t)$ be a biregular curve and have unit speed. Then there is a unique moving frame $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ such that $\vec{\alpha}'(t) = \vec{T}(t)$, and $(T', N', B') = (T, N, B) \begin{bmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} = \begin{cases} T' = k * N \\ N' = -kT + \tau B \\ B' = -\tau N \end{cases}$ [Frenet equation w/ first orthonormal basis vector being the velocity].*

Remark. We say (T,N,B) is the Frenet frame of α and τ is the **torsion** of α. k is clearly the **curvature** of α. Using the Frenet equations.. $\tau = -B' \cdot N = B \cdot N'$ [taken from $N \cdot (B' = -\tau N) \cdot N]$ where $N \cdot N' = 1$ from the properties of an orthonormal basis. We can usually compute the curvature without a Frenet Frame but kind of need it to compute the torsion. **Ex.** The Helix $\alpha(s) = \begin{bmatrix} cos(s/\sqrt{2}) \\ sin(s/\sqrt{2}) \\ \frac{s}{\sqrt{2}} \end{bmatrix}$ has unit speed. Then, $T'(s) = \alpha''(s) = \frac{1}{\sqrt{2}} \begin{bmatrix} -sin(\frac{s}{\sqrt{2}}) \\ cos(\frac{s}{\sqrt{2}}) \\ 0 \end{bmatrix} \Rightarrow N = -1 * \begin{bmatrix} cos(s/\sqrt{2}) \\ sin(s/\sqrt{2}) \\ 0 \end{bmatrix}$.

Note that we need the $-1*$ out in front so that k is positive – there's no such thing as "negative" curvature. N is a unit vector? (must be – part of the orthonormal basis). Similarly [T, N, B] is a moving frame $\iff$ $[T, N, B]$ is an orthonormal basis $\Rightarrow ||T|| = ||N|| = ||B|| = 1$

$\Rightarrow k(s) = \frac{1}{2}$ and $B = TxN = \frac{1}{\sqrt{2}} \begin{bmatrix} sin(s/\sqrt{2}) \\ -cos(s/\sqrt{2}) \\ 1 \end{bmatrix}$, and $\tau = N' \cdot B = \frac{1}{2}$.

Def.

- $< T, N >$ is the plane spanned by $\vec{T}$ and $\vec{N}$ and is termed the oscillating plane (relates to torsion: how quickly this plan changes as we move along the helix).
- $< T, B >$ is the rectifying plane.
- $< N, B >$ is the normal plane.

Question. For a non-unit speed curve $\alpha(t)$, how to compute the Frenet frame, k, and τ?

$\alpha'(t) \xrightarrow{normalize} T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} \xrightarrow{differentiate} N(t) = \frac{T'(t)}{|T'(t)|}$

$N(t) = \frac{T'(t)}{|T'(t)|} \rightarrow B(t) = \vec{T}(t) x \vec{N}(t)$

Theorem 2.4 *For a non-unit speed curve* $\alpha(t)$, $(T', N', B') = (T, N, B) \begin{bmatrix} 0 & -vk & 0 \\ vk & 0 & -v\tau \\ 0 & v\tau & 0 \end{bmatrix}$, *where* $v = |\alpha'(t)|$. *In particular,* $T' = vkN$ *and* $B' = -v\tau N$.

Proof: Let $s(t)$ be an arclength parameterization of α. Define $B(s) := \alpha(t(s))$ has arclength parameterization where $t(s)$ is defined to be the inverse of $s(t)$; $t(s) = (s(t))^{-1}$. Then...

- $T = B'(s) = \frac{\alpha'(t)}{v(t)}$ and $\frac{d}{dt}\alpha(t(s)) = \alpha' * t' = \alpha' * \frac{1}{s'}$.
- $T' = B''(s) = \frac{d}{ds}[\frac{\alpha'(t)}{v(t)}] = \frac{d}{ds}[\frac{\alpha'(t)}{v(t)}] * t'(s) = \frac{\alpha''(t)v(t) - \alpha'(t)v'(t)}{v^3(t)} = k * N(s)$ [last equality due to the Frenet Equation].

- $T'(t) * \frac{dt}{ds} = k * N(s) = k * N(t)$
- $T'(t) = s'(t) * k * N = v * k * N.$

Ex. $\alpha(t) = \begin{bmatrix} e^t cos(t) \\ e^t sin(t) \\ e^t \end{bmatrix}$. Find the Frenet Frame, k, and τ.

$\alpha'(t) = e^t \begin{bmatrix} \sqrt(2)cos(t+\pi/4) \\ \sqrt{2}sin(t+\pi/4) \\ 1 \end{bmatrix}$ and $|\alpha'(t)| = e^t\sqrt{3}$. We've proved $T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt(2)cos(t+\pi/4) \\ \sqrt{2}sin(t+\pi/4) \\ 1 \end{bmatrix}$.

Then $\alpha''(t) = T'(t) = \sqrt{\frac{2}{3}} \begin{bmatrix} -sin(t+\pi/4) \\ cos(t+\pi/4) \\ 0 \end{bmatrix} \Rightarrow N(t) = \begin{bmatrix} -sin(t+\pi/4) \\ cos(t+\pi/4) \\ 0 \end{bmatrix}$, and we can simply write $T'(t) = \sqrt{\frac{2}{3}}N(t)$. And by construction, $\sqrt{\frac{2}{3}} = v(t) * k(t) \Rightarrow k(t) = \frac{\sqrt{\frac{2}{3}}}{e^t * \sqrt{3}} = \sqrt{\frac{2}{3}}e^{-t}$.

Now, we can finally solve for B: $B = TxN = \frac{1}{\sqrt{3}} \begin{bmatrix} -cos(t+\pi/4) \\ -sin(t+\pi/4) \\ \sqrt{2} \end{bmatrix}$ and $B' = \frac{1}{\sqrt{3}} \begin{bmatrix} sin(t+\pi/4) \\ -cos(t+\pi/4) \\ 0 \end{bmatrix}$ using $B' = -v\tau N \Rightarrow \frac{1}{\sqrt{3}} = v\tau \Rightarrow \tau = \frac{1}{\sqrt{3}}v = \frac{1}{3}e^{-t}$.

3 October 4

3.1 Review of Week 1

Def.

- Let $\alpha(t) : \mathbb{R} \to \mathbb{R}^3$ is a space curve which maps a parameter t to a vector in $\mathbb{R}^3$ where $t \in \mathcal{I}$ (where $\mathcal{I}$ is the interval of our domain).
- If $\alpha'(t) \neq 0$ for all t in domain (i.e. $\alpha(t)$) is regular, and we can find an arclength parameterization $\alpha(s)$ such that $|\alpha'(s)| = 1$ (unit speed) for this curve.
- We define $T(s) = \alpha'(s)$ to be the unit tangent vector.
- We define $T'(s) = \alpha''(s)$. If $T'(s) \neq 0, \forall s \in \mathcal{I}$, then the curvature is not ever 0, and the curve is **"biregular"**.
- Note two critical properties of T(s) (i.e. a unit vector part of an orthonormal basis).
  1. $|T(s)|^2 = 1$

2. $|T(s) \cdot T'(s)| = 1$

- We define the "normal vector" to the curve N(s) as the unit vector such that $T'(s) = \kappa N(s)$ where κ is the norm of $T'(s)$ so that $N(s)$ has unit length.
- $\kappa(s) = |T'(s)|$ is the curvature.
- We define the "binormal vector" as $B(s) = T(s)xN(s)$. Together $[T(s), N(s), B(s)]$ form an orthonormal basis $\forall s \in \mathcal{I}$.
- We call $[T(s), N(s), B(s)]$ a "frame" (Frenet Frame), and $[T'(s), N'(s), B'(s)] = [T(s), N(s), B(s)] \begin{bmatrix} 0 & -k & 0 \\ k & 0 & -\theta \\ 0 & \theta & 0 \end{bmatrix}$, which we define as the "Frenet Equation." This equation allows us to solve for N, B, κ, θ given only T.
- θ is called the "torsion" of the curve (tendency to move out of the plane spanned by N and T).
- We can solve for θ: $N' = -kT + B\theta \Rightarrow B \cdot N' = -\kappa B \cdot T + \theta B \cdot B \Rightarrow \theta = B \cdot N'$. Alternatively, we know $B' = -\theta N \Rightarrow B' \cdot (-1) * N = -\theta N \cdot (-1) * N \Rightarrow \theta = -B'N$.

3.2 Helix, Torsion, and Planar Curves

Recall. Helix $\alpha(s) = [\cos(s/\sqrt{2}), \sin(s/\sqrt{2}, -s/\sqrt{2}]$, $\kappa(s) = 1/2$, $\theta(s) = 1/2$.
Ex. Generalized Helix: $\alpha(s) = [r \cos \frac{s}{\sqrt{r^2+h^2}}, r \sin \frac{s}{\sqrt{r^2+h^2}}, h \frac{s}{r^2+h^2}]$ where $r > 0$, $h \in \mathbb{R}$, and arc-length parameterized. Then

1. $\kappa(s) = \frac{r}{r^2+h^2}$ is a constant such that $\kappa(s) > 0$.
2. $\theta(s) = \frac{h}{r^2+h^2}$ is a constant such that $\theta(s)$ can be any arbitrary value.

Proposition 3.0.1 $\theta \equiv 0\ \forall s \iff \alpha(s)$ *is contained within a plane.*

1. θ measures how the oscullating plane moves away from itself.
2. if $\theta \equiv 0$, oscullating plane stays in itself, and $\alpha(s)$ is a planar curve (i.e. convined to a plane).
3. we need to show that $\alpha(t)$ stays within the oscullating plane.

Proof $\Leftarrow$ $\alpha(s)$ is contained within a plane $\Rightarrow T(s)$ and $N(s)$ in the same plane (i.e. form the plane) $\forall s$.
$\Rightarrow B(s)$ is a constant vector since it will always be $\perp$ to the plane spanned by T and N. $\Rightarrow$ B'(s) = 0 $\Rightarrow \theta = 0$.
$\Rightarrow B'(s) = -\theta N(s) = 0 \Rightarrow B(s)$ is a constant. $\Rightarrow$ T(s)N(s) is in the same plane orthogonal to B(s) [otherwise, B(s) wouldn't be a constant] $\Rightarrow \alpha(s)$ is in the same plane.

We could also say $(\alpha \cdot B)'$ [note this is 0 if $\alpha \perp B$] $= \alpha' B + \alpha B' = 0 \Rightarrow \alpha \in B^{\perp}$ [Using $\int_{t_0}^{t} (\alpha \cdot B)' dt \Rightarrow \int_{t_0}^{t} 0 dt$]

Proposition 3.0.2 *If $\alpha(s)$ is planar, and $\kappa(s) = k \neq 0$ (i.e. curvature is a constant), then $\alpha(s)$ is contained in a circle of radius $\frac{1}{k}$. Assume $\alpha(s)$ has arclength parameterization.*

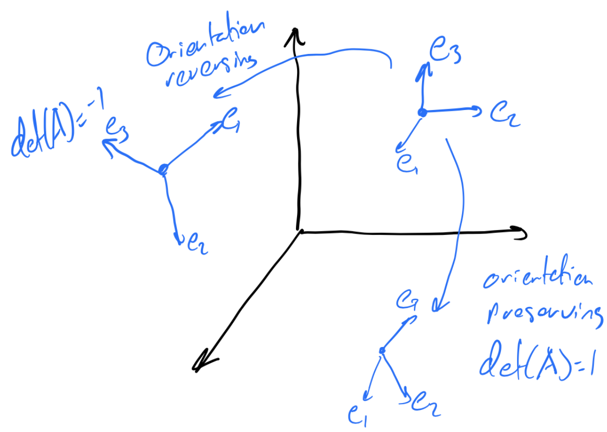
Proof: Let $B(s) = \alpha(s) + \frac{N(s)}{\kappa}$, $B'(s) = \alpha'(s) + \frac{N'(s)}{\kappa} = T(s) + \frac{N'(s)}{\kappa(s)}$ [Recall $N' + T = -\kappa(s)$]. We'd like to show B(s) is a constant.
$\Rightarrow B'(s) = T(s) - T(s)$, so $B(s)$ is a constant; $B(s) = P = \alpha(s) + \frac{N(s)}{\kappa}$, and $|\alpha(s) - P| = |\frac{N(s)}{\kappa} = \frac{1}{\kappa}$ which is the equation of a circle centered at P of radius $\frac{1}{k}$.

3.3 Unique Determination of Curves and Rigid Motion

Goal: For biregular curve, functions $\kappa(s)$ and $\theta(s)$ uniquely determine $\alpha(s)$ up to rigid motion (isometry): $\vec{X} \rightarrow A \cdot \vec{x} + b$.

1. $\vec{X}$ is a point $x \in \mathbb{R}^3$.
2. $A \cdot \vec{x} + b$ is called "rigid motion" if A is an orthogonal matrix and det(A) = +1. b is a constant vector.

Remark. If det(A) = -1, then the orientation of the space is reversed. If det(A) = 1, then orientation is preserved, and this is called "orientation preserving". One can tell orientation is reversed by looking at the basis vectors (and if the right-hand rule is consistent).



Theorem 3.1 *Given differentiable functions $\kappa(s), \theta(s), \kappa(s) > 0$, and $\theta(s) : \mathcal{I} \to \mathbb{R}^3$, $\exists$ a unique biregular curve [up to rigid motion] $\alpha(s) : \mathcal{I} \to \mathbb{R}$, arclength parameterized, with curvature equal to $\kappa(s)$ and torsion equal to $\theta(s)$.*

1. Easy to see that rigid motion preserves $\kappa(s)$, $\theta(s)$ [invariant to rigid motion transformations].

2. Not too hard to show that given $\alpha(s), \bar{\alpha}(s)$ with $\kappa(s) = \bar{\kappa}(s)$ and $\theta(s) = \bar{\theta}(s)$, then $\exists$ a rigid motion that takes α to $\bar{\alpha}(s)$. The question becomes how to map this rigid motion.

   (a) $\alpha(0) \to \bar{\alpha}(0)$ (i.e. map a certain point from α to $\bar{\alpha}$.

   (b) $[T(0), N(0), B(0)] \to [\bar{T}(0), \bar{N}(0), \bar{B}(0)]$ (i.e. map the Frenet frame at that point). It's not affected by the affine transformation A.

3. Harder to show the existence of $\alpha(s)$ given $\kappa(s), \theta(s)$. Given a $\kappa(s)$ and a $\theta(s)$, if we can construct a frame with these values, we would be able to find a curve with the specified curvature and torsion.

4. We would need to solve for a moving frame [T(s), N(s), B(s)] such that $[T', N', B'] = [T, N, B] \begin{bmatrix} 0 & -k & 0 \\ k & 0 & -\theta \\ 0 & \theta & 0 \end{bmatrix}$. We call $F(s) = [T, N, B]$ and $W(s) = \begin{bmatrix} 0 & -k & 0 \\ k & 0 & -\theta \\ 0 & \theta & 0 \end{bmatrix}$.

   Once we solve this equation, we can get [T, N, B], and can integrate T to get α.

Proof: $F'(s) = F(s)W(s)$, so existence of $\alpha(s)$ follows from the existence of solutions to the second order, LTI ODE. Note $W^T = -W$. Then we have $(FF^T)' = F'F^T = F(F')^T = FWF' + F(FW)^T$. Note: If A is orthogonal, $A * A^T = A^T A = I$. Then we have, $= FWF^T + FW^T F^T = F(W + W^T)F^T = 0$. So FF^T must be a constant.

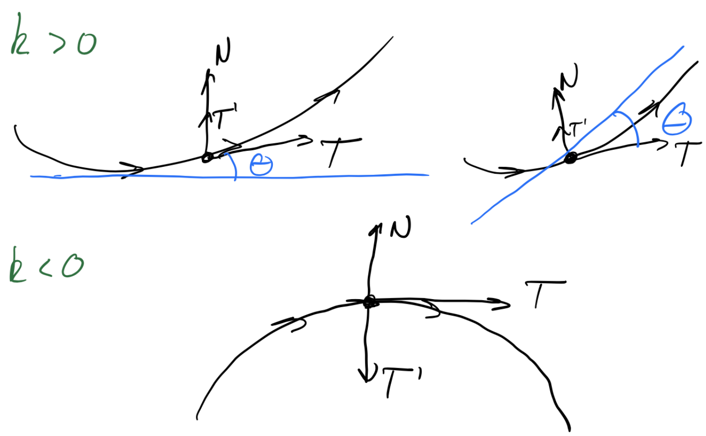
We now proceed with the mapping of a point (i.e. 0). $F(0)F^T(0) = I \iff$ we choose $T(0), N(0), B(0)$ to be an orthonormal basis $\Rightarrow F(s)F^T(s) = I$.

Note, we have the freedom to choose the base point that we map by the rigid motion transformation as well as the direction of the curve (i.e. we can invert the direction) that has the same curvature and torsion.

This is somewhat remarkable. We simply are given a curvature, and torsion value, we choose a base point, and we will get a biregular curve going through this base point. We get the rest of the curve, along with the Frenet frame at this base point from following the above procedure [integrate T(s) to get $\alpha(s)$]. Solving the LTI, second order ODE, and setting its initial conditions sets the base point of the curve.

Corollary 3.1.1 *$\alpha(s) : \mathcal{I} \to \mathbb{R}^3$, arclength param, has $\kappa(s) = \kappa$, $\theta(s) = \theta$ for $s \in \mathcal{I}$, then $\alpha(s)$ must be the generalized helix up to a rigid motion transformation.*

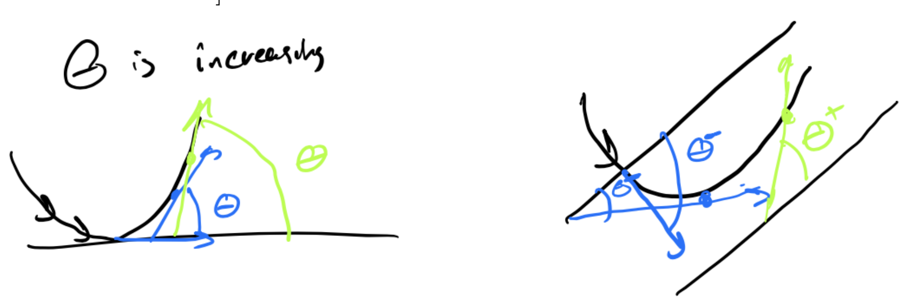
Remark: In the case of planar curves, there is another natural choice of $N(s)$, namely to rotate $T(s)$ counter clockwise by $90°$. Then $T'(s) = \kappa(s)N(s)$, and this choice of formulation allow $\kappa(s)$ to have a negative sign.



Note: $\kappa > 0$ in the first image because T' is increasing $\iff$ points inside the curve (i.e. T is increasing). $\kappa > 0 \iff$ N and T' point in the same direction. In these cases, the polar angle θ always increases [independent of which axis you choose to define θ].

Assume $T(s) = [\cos\theta(s), \sin\theta(s)]$, then T'(s) $= \theta'(s)[-\sin\theta(s), \cos\theta(s)] = \theta'(s)N(s)$. So $\kappa(s) = \theta'(s) \Rightarrow$ how θ changes with the curve (is the angle increasing or decreasing)?

In this view, the geometric meaning of $\kappa(s)$ for a planar curve is the speed at which the angle of T changes. [independent of whichever direction (or axis) you fix to measure θ].



4 October 6

4.1 Directional Derivatives, Vector Fields

Def. Directional Derivatives

- $f : U \subset \mathbb{R}^n \to \mathbb{R}$, differentiable at any point $\forall p \in U$ where U is an open set (open set is the generalization of the "open interval" to a space.

- $\frac{\delta f}{\delta x_i}|_p$ or the partial of f with respect to x_i at point p = $\lim\limits_{h \to 0} \frac{f(p+h*x_i)-f(p)}{h}$.
- e.g. $f(x_1, ..., x_n)$, $\frac{\delta f}{\delta x_1}|_{[a_1,a_2,...,a_n]} = \lim h \to 0 \frac{f(a_1+hx_1,a_2,...,a_n)-f(a_1,...,a_n)}{h}$.
- More generally, for an arbitrary vector $v \in \mathbb{R}^n$, $\lim\limits_{h \to 0} \frac{f(p+hv)-f(p)}{h}$ is the derivative of f at point p in the direction of v = $D_v f|_p$.
- Assume v = $(v_1, v_2, ..., v_n)^T$, then $D_v f|_p = \sum\limits_{i=1}^{n} v_i \frac{\delta f}{\delta x} = v \cdot \nabla f|_p$, where $\nabla f|_p$ is the gradient of f at point p; $\nabla f =< \frac{\delta f}{\delta x_1}, .., \frac{\delta f}{\delta x_n} >$
- We can view the derivatives of f at p as a linear map: $Df|_p : \mathbb{R}^n \to \mathbb{R}$, and carry this definition over to different directions v: $< v_1, v_2, ...v_n >^T =$ $v \to D_v f|_p = \sum\limits_{i=1}^{n} v_i \frac{\delta f}{\delta x_i}$.

Def. Vector Fields

- Now we can allow v to vary with $p \in U$ (i.e. $v = \begin{bmatrix} 3x \\ x+y \\ 3 \end{bmatrix}$ (i.e. can change smoothly w.r.t. changes in p) to get a scalar function $D_v f$ that varies with p [this will be the directional derivative of how f changes with respect to the directional field defined by v].
- We say v is a **vector field**. f is simply a function that maps an element of $\mathbb{R}^n \to \mathbb{R}$. Later, we will call f a **1-form**.
- **Directional derivative of function f along a given vector v** (at a specific point x): $D_v f(x) = \sum\limits_{i=1}^{n} v_i \frac{\delta f}{\delta x_i}$ and the general field [describes this function for all points] is denoted by $v[f]$.
- **the following notation is equivalent**: $\nabla_v f = D_v f = v \cdot \nabla f = v[f]$.
  - $v[af + bg], a, b \in \mathbb{R}$, f, y are functions operating on U, v is a vector field (i.e. unique vector for each point $p \in U$), then $v[af + bg] = av[f] + bv[g]$.

– In other words, we take the dot product between the vector field v and the vector mapping $f, y : \mathbb{R}^n \to \mathbb{R}^n$, and compute the dot product of the gradient field of $[af + bg]$ with the vector field defined by v.

Theorem 4.1 (Leibniz Rule) *$v[f*g] = v[f]*g+f*v[g]$ (i.e. $v[$product of two functions$]$).*

Proof:

$$\begin{aligned}
D_v f = v[fg] &= \sum_{i=1}^{n} v_i \frac{\delta}{\delta x_i}(fg) \\
&= \sum_{i=1}^{n} v_i * (\frac{\delta f}{\delta x_i} * g + f * \frac{\delta g}{\delta x_i}) \\
&= \sum_{i=1}^{n} v_i (\frac{\delta f}{\delta x_i}) * g + \sum_{i=1}^{n} v_i f (\frac{\delta g}{\delta x_i}) \\
&= v[f]g + fv[g]
\end{aligned}$$

This is clearly a scalar output: $v[f]$ is the directional derivative of function f in the direction of v, and g is a function that maps a point to a scalar value. Therefore at all pointsd, this will be a single scalar value.

4.2 Tangent Space

Def.[Tangent Space of p = vectors based at p] is $T_p\mathbb{R}^n \cong \mathbb{R}^n$ (i.e. the tangent space is isomprphic – like a structure-preserving bijection). It's the space of vectors tangent to a given point on a surface.

Notation:

- $\frac{\partial}{\partial x_i}$ is the ith orthonormal basis vector spanning a n-dimensional space. For example in $\mathbb{R}^3$, $\frac{\partial}{\partial x_1} = [1, 0, 0]$, $\frac{\partial}{\partial x_2} = [0, 1, 0]$, and $\frac{\partial}{\partial x_3} = [0, 0, 1]$.

Ex.

- Let $v = [v_1, v_2, ..., v_n]^T \in T_p\mathbb{R}^n$ be a vector in the tangent space of point p. Then $v = \sum\limits_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$.
- Then the directional derivative of f in the direction of v at a point p is simply denoted by $v[f] = \nabla_v f = \sum\limits_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}$
- i.e. $v = \{1, 0, 0, ...0 >^T \in T_p\mathbb{R}^n$, then $v = \frac{\partial}{\partial x_1}$.

- Let $v = x_1 \frac{\partial}{x_1} + 2x_2 \frac{\partial}{\partial x_2}$ be a vector field and $f(x_1, x_2) = x_1 x_2^2$, $\nabla f =< x_2^2, 2x_1 x_2 >^T =< \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} >^T$ be a function on our space. Then the directional derivative of f in the direction specified by the vector field is:

$$\begin{aligned} v[f] = \nabla_v f &= x_1 \frac{\partial f}{\partial x_1} + 2x_2 \frac{\partial f}{\partial x_2} \\ &= x_1 * x_2^2 + 2x_2 * 2x_1 * x_2 \\ &= 5x_1 x_2^2 \end{aligned}$$

4.3 Co-Tangent Space and 1-forms

Def.[Cotangent Space].
The co-tangent space of a point in the tangent plane, denoting the co-tangent space at a point p as $T_p^*\mathbb{R}^n$ is the set of linear functions (also called "1-forms") that map a vector from the tangent space to $\mathbb{R}$: $\{$Linear functions: $\alpha | T_p\mathbb{R}^n \to \mathbb{R}\}$.

Background[Dual Space]

- Any vector space V has a corresponding dual vector space (or dual space for short).
- The dual space consists of all linear forms on V. A linear form is a linear function that maps an element of V to an element of $\mathbb{R}$.
- A **one form** takes **one** vector from V and maps it to $\mathbb{R}$
- A **zero form** takes **zero** vectors from V and maps it to $\mathbb{R}$ (this is the set of reals mapping back onto themselves).
- In general, a *k-form* on a vector space V over a Field F that maps *k* vectors from V to a scalar (i.e. an element of F).
- Typically the cotangent space is defined as the "dual" of the tangent space.

Def.[1-form]
Denote the co-tangent space of a point p as $T_p^*\mathbb{R}^n$. Than an element of this set is called a "1-form": $\alpha | T_p\mathbb{R}^n \to \mathbb{R}$. This mapping is specified by the basis vectors at the point p: $\{\frac{\partial}{\partial x_i}\}|_p$.

You've seen this before: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is a **one form** where $dx = \frac{\partial}{\partial x}$ and $dy = \frac{\partial}{\partial y}$ are the basis vectors in the x and y directions respectively and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ is the linear combination of the basis vectors at point p.

Summary with Examples

- **tldr:** The co-tangent space is the set of linear functions α (or 1-forms) that map elements of our tangent space back into the set of reals.

- For 1-form α, $\alpha(\sum\limits_{i=1}^{n} v_i \frac{\partial}{\partial x_i}) = \sum_{i=1}^{n} v_i \alpha(\frac{\partial}{\partial x_i})$.

  Note: α acts on the basis vectors $\frac{\partial}{\partial x_i}$.

- To formalize the mapping of independent dimensions from the tangent space to a scalar, we denote $\{dx_i\}$ as the **cotangent vector**.

  $dx_i(\frac{\partial}{\partial x_i}) = \delta_{ij}$ [remember $\delta_{ij} = 1 \iff i = j$, otherwise 0].

- We say $\{dx_i\}$ are the dual of $\{\frac{\partial}{\partial x_i}\}$ as $\{dx_i\}$ maps $\frac{\partial}{\partial x_i}$ to an element of the reals.

- e.g. $dx_1(\sum\limits_{i=1}^{n} v_i \frac{\partial}{\partial x_i}) = v_1$.

Theorem 4.2 *Any $\vec{\alpha} = \sum_{j=1}^{n} \alpha_j dx_j$ is in the co-tangent space.*

Proof:

$$\begin{aligned}
\alpha(\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}) &= \sum_{i=1}^{n} \alpha v_i (\frac{\partial}{\partial x_i}) \\
&= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} \alpha_j dx_j \frac{partial}{\partial x_i} \\
&= \sum_{j=i}^{n} v_i \sum_{j=1}^{n} \alpha_j \delta_{ij} \\
&= \sum_{i=1}^{n} v_i \alpha_i
\end{aligned}$$

So in summary we have $\vec{\alpha} = \sum_{j=1}^{n} \alpha_j dx_j$ [a linear function] and $\sum\limits_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$ [vector in tangent space] $= \sum_{i=1}^{n} v_i \alpha_i$ [element of $\mathbb{R}$]. So we have proved $\vec{\alpha}$ is a 1-form in our co-tangent space.

Def.[Differential 1-form]

Let $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ ... \\ \alpha_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$, and $\alpha(v) = \begin{bmatrix} \alpha_1, \alpha_2, ... \alpha_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$.

If we allow $\vec{\alpha}$ to vary with p – i.e. at each point, we pick a linear function – then $\vec{\alpha}$ is called a "differential 1-form". This is the same as a normal 1-form, but now the mapping varies depending on which "tangent space" we're at.

More formally, $\vec{\alpha}|_p \in T^*_p\mathbb{R}^n$.

Ex.

- $\alpha = x_2dx_1 + 3x_1dx_2$ is a differentiable 1-form.
- $v = (x_1+x_2)\frac{\partial}{\partial x_1} + x_1x_2^2\frac{\partial}{\partial x_2}$ be a vector field [defined at each point on $\mathbb{R}^2$]. Clearly $v \in T_p\mathbb{R}^2$ as v is tangent to $\mathbb{R}^2$ at each point.
- $\alpha(v)$ is a "smooth 1-form": it is a smooth function that outputs a scalar at each point.

  $$\begin{bmatrix} x_2, 3x_1 \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ x_1x_2^2 \end{bmatrix} = x_1x_2 + x_2^2 + 3x_1x_2^2$$

- $\alpha(v)|_{(0,0)} = \alpha|_{(0,0)}(v|_{(0,0)}) = 0$.
  - $\alpha|_{(0,0)}$ restricts the differential 1-form to the value at point p = (0,0).
  - $(v|_{(0,0)})$ selects the vector from the vector field at point p = (0,0).

4.4 Line Integrals

Define $\gamma : \mathcal{I} \to \mathbb{R}^n$ to be a smooth curve $\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ ... \\ \gamma_n(t) \end{bmatrix}$ with tangent vector $\begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ ... \\ \gamma_n'(t) \end{bmatrix} \in T_{\gamma(t)}\mathbb{R}^n$ [i.e. the vector of partials of $\gamma(t)$ defines the tangent space along curve $\gamma(t)$].

We can define a "1-form" along γ: $\alpha = \sum\limits_{i=1}^{n} \alpha_i(\gamma_1, \gamma_2, ..., \gamma_n)dx_i|_{\gamma(t)}$.

1. α is a differentiable 1-form: therefore, it depends on the **location** of the point p.
2. If we restrict α to run along γ, then we need to use the parameterization of γ when computing the transformation induced by α. This is denoted by $\alpha(\gamma_1, \gamma_2, ..., \gamma_n)$.

Taking this a step further, we can integrate the 1-form along γ:

$$\begin{aligned} \int_\gamma \alpha &= \sum_{i=1}^{n} \int_a^b \alpha_i(t)\gamma_i'(t)dt \\ &= \text{sum along all n-dimensions integrating along the curve} \\ &= \int_a^b \alpha(\gamma'(t))dt \end{aligned}$$

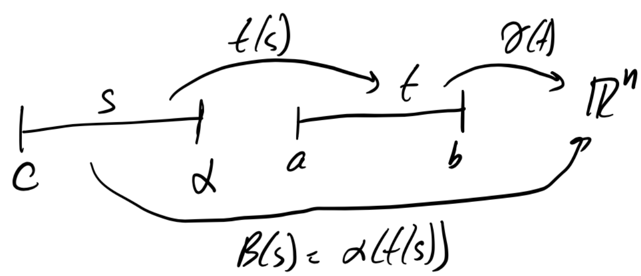
Ex.

$\gamma(t) =< t^2, t^3 >$, $t \in (0,1]$ is our curve **parameterized with t**, $\alpha = dx_1 - \frac{1}{x_1}dx_2$ is our 1-form **parameterized by** x_1, x_2. We therefore need to substitute $< t^2, t^3 >$ into our 1-form to find the linear combination factors along the points of our curve.

We'd like to compute the integral of α along γ which is equivalent to $\int_\gamma \alpha$. Then:

- $\gamma'(t) =< 2t, 3t^2 >$
- $\alpha|_{\gamma(t)} = dx_1 - \frac{1}{t^2}dx_2$.
- **Recall** dx_i is the cotangent vector such that $dx_i(\frac{\partial}{\partial x_i}) = \delta_{(ij)}$. You can think of $\frac{\partial}{x_i}$ as the ith orthonormal basis basis component for how a function changes with respect to the ith dimension. dx_i is simply an element of the dual that maps this vector to 1 $\iff i = j$, otherwise 0.
- Then $\alpha(\gamma'(t)) = 2t - 3\frac{t^2}{t^2} = 2t - 3$.
- And $\int_\gamma \alpha = \int_a^b \alpha(\gamma'(t)) = \int_0^1 2t - 3dt = t^2 - 3t|_0^1 = -2$ which is the integral of this 1-form along our curve $\gamma(t)$.

Note: The integral is independent of parameterization!



Theorem 4.3 *The integral along a curve is independent of your choice of parameterizations:* $\int_\beta \alpha = \int_\gamma \alpha$ *[orientation does matter though – up to a difference in sign].*

Proof: Assume β, γ have the same orientation, $t(s)$ is increasing. Then $\frac{B}{ds} =$

$\frac{d\gamma}{dt} * t'(s)$ and

$$\begin{aligned}
\int_\beta \alpha &= \int_c^d \alpha(\frac{d\beta}{ds})ds \\
&= \int_c^d \alpha(\frac{d\gamma}{dt} * t'(s))ds \\
&= \int_c^d t'(s)\alpha(\frac{d\gamma}{dt})ds \text{ [$t'(s)$ is a scalar in a linear fn. -- can pull out]} \\
&= \int_a^b \alpha(\frac{d\gamma}{dt})dt \\
&= \int_\gamma \alpha
\end{aligned}$$

Def.[Special 1-form]:
A special 1-form is when $\alpha = df$ for some function f. Recall this from 18.02. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Then if $\alpha = df$, $\alpha = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + ...$ where $a_j = \frac{\partial f}{\partial x_j}$ and dx is simply the cotangent vector that maps $\frac{\partial}{\partial x}$ to 1. We say df acts on an element of the tangent space by $df(\frac{\partial}{\partial x_i}) = \frac{\partial f}{\partial x_i}$.

- Hence, $df(v) = df(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i} = \nabla_f \cdot [dx_1, dx_2, ..., dx_n]$
- Simply, $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} * dx_i$ for some function f.

Theorem 4.4 $\int_\gamma df = f(b) - f(a)$ *[only depends on the endpoints].*

$\gamma : [a, b] \to \mathbb{R}^n$ [left as an exercise].
Def. 1-form α is called exact if $\alpha = df$, f : function.

5 October 11

We now shift our focus from curves in $\mathbb{R}^2$ to surfaces in $\mathbb{R}^3$. Notably, the distance between two points when traveling on a surface as well as the curvature and torsion of surfaces. in $\mathbb{R}^3$.

5.1 Surfaces in $\mathbb{R}^3$

Def.[Exterior Derivative]
Also known as the "differential". This is simply how a function changes with respect to changes along basis vectors that span the tangent space at a point.

Def.[Surface]

A surface is $\alpha : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ [U is an open subset of $\mathbb{R}^2$], and we assume α is smooth (i.e. differentiates).

We can now define our notion of **regularity** on a surface.
Let (u,v) parameterize the open subset U. Then α maps points of U onto a surface in $\mathbb{R}^3$. We say that surface is **regular** if $\frac{\partial\alpha}{\partial u}$ and $\frac{\partial\alpha}{\partial v}$ are linearly independent at each point.

Note As we now live in $\mathbb{R}^2$, it takes two basis vectors to fully describe the tangent space (i.e. the tangent plane) at a point p. If these vectors are not independent, then the tangent plane collapses to a line, and and the surface is not regular.

Preliminaries.
$\alpha(u,v) = \begin{bmatrix} \alpha_1(u,v) \\ a_2(u,v) \\ a_3(u,v) \end{bmatrix}$ is a surface which maps $U \subset \mathbb{R}^2 \to \mathbb{R}^3$. We denote $\frac{\partial\alpha}{\partial u}$ as α_u and $\frac{\partial\alpha}{\partial v}$ as α_v.

Ex.
The tangent space to a point on our surface at (u,v) is spanned by: $\frac{\partial\alpha}{\partial u} = \begin{bmatrix} \frac{\partial\alpha_1}{u}(u,v) \\ \frac{\partial\alpha_1}{u}(u,v) \\ \frac{\partial\alpha_1}{u}(u,v) \end{bmatrix}$ and $\frac{\partial\alpha}{\partial v} = \begin{bmatrix} \frac{\partial\alpha_1}{v}(u,v) \\ \frac{\partial\alpha_1}{v}(u,v) \\ \frac{\partial\alpha_1}{v}(u,v) \end{bmatrix}$.
We call the 2-dimensional plane spanned by α_u, α_v the tangent plane at $\alpha(u,v)$.

Def.[Derivative Along a Surface]
The derivative of the surface α is $D\alpha : \mathbb{R}^2 \to \mathbb{R}^3$. Simply, it's a linear map represented by the matrix of basis vectors spanning the tangent plane: (α_u, α_v) $= \begin{bmatrix} \frac{\partial\alpha_1}{\partial u}, \frac{\partial\alpha_1}{\partial v} \\ \frac{\partial\alpha_2}{\partial u}, \frac{\partial\alpha_2}{\partial v} \\ \frac{\partial\alpha_3}{\partial u}, \frac{\partial\alpha_3}{\partial v} \end{bmatrix}$ and is a linear approximation for how a two dimensional input in the domain $\begin{bmatrix} u_i \\ v_i \end{bmatrix}$ will cause the output of the function to change.

Def.[Directional Derivative Along a Surface]
For any $w \in \mathbb{R}^2$, $D_w\alpha = \begin{bmatrix} D_w\alpha_1 \\ D_w\alpha_2 \\ D_w\alpha_3 \end{bmatrix}$. This is the directional derivative of α in the direction along w. $D_w\alpha = D\alpha(w)$. It representes how each component would change by moving in the $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ direction.

Ex.

Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1 \frac{\partial}{\partial u} + w_2 \frac{\partial}{\partial v}$. Then $D\alpha(w) = [a_u, a_v] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1 \alpha_u + w_2 \alpha_v =$ scalar value.

What is the image of $D\alpha$? The input dimensionality is $\mathbb{R}^2$, but what space does the output belong to? $D\alpha$ maps points to $\mathbb{R}^3$. $\text{Im}(D\alpha)$ = the plane spanned by $[\alpha_u, \alpha_v]$ (i.e. the tangent plane's basis vectors, each of which live in $\mathbb{R}^3$).

Ex.

1. Let $S^2 = \begin{bmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{bmatrix}$, $u \in (0, \pi)$, $0 < v < 2\pi$.

   Left as an exercise: **Check** this is a regular surface. **Recall** a regular surface is one in which the tangent plane at each point is well-defined. You can verify this by computing two tangent vectors at each point on the surface and showing that they are linearly independent.

   We would need to exclude certain points to make this surface regular (non-linearly independent tangent vectors).

   Note: (u, v) can take on all real numbers, but this parameterization [along with $u \in (0, \pi$, $0 < v < 2\pi)$ will result in (u, v) missing the longitude of the sphere [tangent vector is 0 along this curve $\Rightarrow$ not regular].

   One cannot find a parameterization of a sphere that covers every point.

2. $f : U \subset \mathbb{R}^2 \to R$. $\alpha(u, v) = \begin{bmatrix} u \\ v \\ f(u, v) \end{bmatrix}$, $\alpha_u = \begin{bmatrix} 1 \\ 0 \\ f_u \end{bmatrix}$, $\alpha_v = \begin{bmatrix} 0 \\ 1 \\ f_v \end{bmatrix}$.

   α_u and α_v are clearly linearly independent, therefore, the graph (i.e. α, synonymous with function) yields a regular surface.

   To rigorously prove linear independence, one would show:

   $a\alpha_u + b\alpha_v = 0 \Rightarrow \begin{bmatrix} a \\ b \\ af_u + bf_v \end{bmatrix} = \vec{0} \iff a, b = 0]$.

3. Surface of Revolution. Denote function f: $\mathcal{I} \to \mathbb{R}$, $f > 0$ [f takes positive values]. Then take the graph $x = f(z)$ in the x, z plane and rotate it about the z-axis. Then we get a "surface of revolution.

   When can we say this is a regular surface?

   Compute the two tangent vectors: $\alpha(u, v) = \begin{bmatrix} f(u) \cos v \\ f(u) \sin v \\ u \end{bmatrix}$, $u \sim z$, $v \sim \theta$

   [circular coordinates r, θ parameterize a revolutory surface].

$$\alpha_u = \begin{bmatrix} f'(u)covv \\ f'(u)sinv \\ 1 \end{bmatrix}, \ \alpha_v = \begin{bmatrix} -\sin v f(u) \\ \cos v f(u) \\ 0 \end{bmatrix}$$

Homework exercise: when this surface is regular? When are these two tangent vectors linearly independent?

4. Tangent Developable. Let $\gamma(s)$ $[\mathcal{I} \to \mathbb{R}^3]$ be a regular curve parameterized by arclength. Define $\alpha(u,v) = \gamma(u) + v \cdot \gamma'(u)$. $u \in \mathcal{I}$. **Remember, (u,v) are the basis vectors of your domain** $\in \mathbb{R}^2$.

   Then $\alpha_u = \gamma'(u) + v\gamma''(u)$ and $\alpha_v = \gamma'(u)$

   $$\begin{cases} u \in \mathcal{I}, v \neq 0 \text{ (is regular)} \\ v = 0 \text{ (not regular)} \end{cases}$$

   It is regular when $v \neq 0$, because then we get a two dimensional vector space. Otherwise when $v = 0$, it collapses down to a single dimension.

5. Implicitly Defined Surfaces. Let f be a 1-form: $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ and consider the level set $S = \{(x_1, x_2, x_3) | f(x_1, x_2, x_3) = 0\}$

   f(x,y,z) = z is a 1-form and S = the x-y plane.

   Note. f(x,y,z) = 0 when $x, y, z \in S$.

   For a different function defined as f(x,y,z) = 0, S = $\{(x,y,z) \in \mathbb{R}^3\} = \mathbb{R}^3$.

5.2 Implicitly Defined Surfaces

Theorem 5.1 *If $\nabla f \neq 0$ at $p \in U \subseteq \mathbb{R}^3$, then we can choose coordinates at p so that level set equals a graph near p [i.e. a function near p]. Allows us to check whether the level set can be a regular surface (check if $\nabla f = 0$ (or not)).*

Proof: Implicit function theorem.

Note: Rather than defining a surface, differentiating with respect to the basis vectors in the domain to find a parameterization of the tangent vectors, and then analyzing if those tangent vectors are linearly independent (or not). If our surface is defined as a level set of a surface in $\mathbb{R}^4$, **we simply look at whether ∇f is $\vec{0}$ or not**. This is because $df = 0$ at a level surface, so we have $-\frac{\partial f}{\partial z}dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. So long as $\nabla f \neq 0$, we'll have linearly independent tangent vectors.

Note: In this case, we can define **one** surface to be the level set of **another** function. Then if $\nabla f|_p \neq 0$, the surface defined by our level set is regular. Note: on a surface in $\mathbb{R}^4$, for a level set, we effectively lose one dimension $\Rightarrow$ surface in $\mathbb{R}^3$ with a tangent plane in $\mathbb{R}^2$. Looking at ∇f will determine if the vectors spanning the tangent plane in $\mathbb{R}^2$ are linearly independent..

Ex. 1
Let f be a function: $f(x,y,z) = x^2 + y^2 - z^2$ and S be the 0-level set of f: $S = f^{-1}(0) = \{(x,y,z)|x^2 + y^2 - z^2 = 0\}$.
We can compute $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x = y = z = 0$.
So $S/\{(0,0,0)\}$ is regular.

Ex. 2
$x^2 + y^2 - z^2 = 0$
$x^2 + y^2 = z^2$
$|z| = \sqrt{x^2 + y^2}$ (also a surface of revolution). In our homework, we will see a torus can also be represented in this way (torus such that the surface is a level set of this function).

Question: What do you mean by level set equals a graph near p?

Def[Unit Normal]
Let α_u, α_v be vectors spannign our tangent space. Then the unit normal vector $= n = \frac{\alpha_u x \alpha_v}{||a_u x \alpha_v||}$.

Note: $\vec{n}$ is orthogonal to α_u, α_v, and (α_u, α_v, n) is positively orientated). n is the normal vector of the tangent plane, so it is independent of the parameterization (up to a sign).

Foreshadowing:
If we can understand how the unit normal changes with the surface, we can understand how the surface is bending or changing at a point.

Theorem 5.2 *For an implicitly defined surface,* $n = \pm\frac{\nabla f}{||\nabla f||}$.

Proof:
$\alpha : (u,v) \to \mathbb{R}^3$ is a regular parameterization of the level set $f^{-1}(0)$.
α_u, α_v : are tangent vectors. Then $f(\alpha(u,v)) = 0$ [$\alpha(u,v)$ is a parameterization of the 0-level set of f $\Rightarrow$ f(any element in this set) = 0].
If we move tangent to the 0-level set... then we remain on the 0-level set. So then, $\nabla f \cdot \alpha_u = 0 = \nabla f \cdot \alpha_v$.
Since $\nabla f \perp \alpha_v$, $\nabla f \perp \alpha_v$, ∇f is normal to α_u and α_v and is therefore a multiple of $\vec{n}$. So $n = \pm\frac{\nabla f}{||\nabla f||}$ [also we learned in 18.02 that the level set is $\perp$ the gradient...].

5.3 First Fundamental Form

Def.[First Fundamental Form]
I is a positive definite, bilinear function on $T_{(u,v)}\mathbb{R}^2$ $w, z \in T_{(u,v)}\mathbb{R}^2$, then $I(w, z) = D\alpha(w) * D\alpha(z)$.

Notation:

- **Recall** $D\alpha(w) = D_w\alpha$ = directional derivative of α along the direction of w.
- bilinear function: a function of two variables that is linear with respect to each of its arguments, separately.

  bilinear form = bilinear function = bilinear map : VxV $\rightarrow$ K

  Vector space V, field K.

I = $\begin{bmatrix} D\alpha(u) \cdot D\alpha(u), D\alpha(u)D\alpha(v) \\ D\alpha(v) \cdot D\alpha(u), D\alpha(v) \cdot D\alpha(v) \end{bmatrix} = \begin{bmatrix} \alpha_u\alpha_u, \alpha_u\alpha_v \\ \alpha_v\alpha_u\alpha_v\alpha_v \end{bmatrix}$
Matrix is clearly symmetric, and is positive definite matrix: $z^T\mathcal{I}z > 0$ for all column vectors z.

Another interpretation: Let u, v be basis vectors in $U \subseteq \mathbb{R}^2$. Then $\alpha(u, v)$ maps each point in U (parameterized by u, v) to a point on the surface $S \in \mathbb{R}^3$. The basis vector u is transformed to the vector $D\alpha(u) = \alpha_u$ which along with α_v spans the tangent plane at a point.

Any vectors in the tangent plane, v, $\vec{t} = (x\alpha_u + y\alpha_v)$ is a linear combination of the basis vectors spanning this plane. Then, we can define the dot product in $\mathbb{R}^3$ confined to the tangent plane spanned by α_u and α_v: for a vector of the tangent plane $\vec{t} = (\vec{t} \cdot \vec{t}) = (x\alpha_u + y\alpha_v) \cdot (x\alpha_u + y\alpha_v) = Ex^2 + 2Fxy + Gy^2$ where $E = \alpha_u \cdot \alpha_u$, $F = \alpha_u \cdot \alpha_v$, $G = \alpha_v \cdot \alpha_v$.

E,F,G clearly depend on the point $P \in S$, and we map a point $r \in U$ to a point $\alpha(r) = p \in S$. So, E, F, G can be viewed as functions on U (i.e. the space spanned by (u,v)). Knowing E, F, G is equivalent too knowing the first fundamental form.

From Wikipedia The first fundamental form is the inner product on the tangent space of a surface in three-dimensional Euclidean space which is induced canonically from the dot product of $\mathbb{R}^3$. It permits the calculation of curvature and metric properties of a surface such as length and area in a manner consistent with the ambient space.

In other words, it lets us calculate curvature and length by approximating these quantities on the tangent plane of a curved surface.

Ex.
Let X(u,v) be a parameteric surface. Then the inner product of two tangent vectors X_u, X_v is:

$$\begin{aligned} I(aX_u + bX_v, cX_u + dX_v) &= ac < X_u, X_v > + (ad + bc) < X_u, X_v > + bd < X_u, X_v > \\ &= Eac + F(ad + bc) + Gbd \end{aligned}$$

Where E, F, G are the coefficients of the first fundamental form. $\mathcal{I}(x, y)$ can also be represented as $x^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} y$.

The first fundamental form completely describes the metric properties of a surface. It enables one to calculate the lengths of curves on the surface and the areas of regions on the surface.

Intuition: We can approximate the distance on a curved surface by computing the tangent plane at each point, computing a small "distance" on this tangent plane, and then moving over by a small amount to the next point on the surface and repeating this process.

Def. [Line Element] The line element (or length element) can informally be thought of as a line segment associated with an infinitesimal displacement vector in a metric space. The length of the line element, which may be though of as a differential arc length, is a function of the metric tensor and is denoted by ds.

Def.[Metric Tensor] A metric tensor (or simply metric) is an additional structure on a manifold M (such as a surface) that allows defining distances an angles, just as the inner product on a Euclidean space allows defining distances and angles there. More precisely, a metric tensor at a point p of M is a bilinear form defined on the tangent space at p (that is, a bilinear function that maps pairs of tangent vectors to real numbers), and a metric tensor on M consists of a metric tensor at each point p of M that varies smoothly with p.

Ex.[Line Element as a fn. of First Fundamental Form]:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

The square of the line element is equal to the first fundamental form [i.e. dot product of two vectors in the tangent space]: $a \cdot b = \frac{\cos(\theta)}{||a||||b||}$

Ex.[Lengths on Manifolds/Surfaces]. The length of a curve on a sphere.

We can parameterize a sphere with $X(u, v) = \begin{bmatrix} \cos u \cos v \\ \sin u \sin v \\ \cos v \end{bmatrix}$ such that $(u, v) \in [0, 2\pi)x[0, \pi]$.

We can find the basis vectors of the tangent space to each point on the sphere: $X_u = \begin{bmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{bmatrix}$ and $X_v = \begin{bmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{bmatrix}$

Therefore, the first fundamental form is found by taking the dot product (i.e. metric tensor) between the basis vectors spanning the tangent space: E = $X_u \cdot X_u = \sin^2 v$, $F = X_u \cdot X_v = 0$, $G = X_v \cdot X_v = 1$. Therefore, $\mathcal{I}(u,v) = \begin{bmatrix} \sin^2 v & 0 \\ 0 & 1 \end{bmatrix}$.

The equator of a unit sphere can be parameterized with $(u(t), v(t)) = (t, \frac{\pi}{2})$ with $0 \leq 0 \leq 2\pi$. Then we may use the line element ds to calculate the length of this curve:

$$\begin{aligned} \int_0^{2\pi} \sqrt{ds^2} &= \int_0^{2\pi} \sqrt{E(\frac{\partial u}{\partial t})^2 + 2F\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + G(\frac{\partial v}{\partial t})^2} dt \\ &= int_0^{2\pi} \sqrt{E1^2 + 2F1 * 0 + G(0)^2} dt \\ &= int_0^{2\pi} |\sin v| dt \\ &= 2\pi \sin(\frac{\pi}{2}) = 2\pi \end{aligned}$$

This comes from the tangent plane being a good approximate to the surface for infinitesimally small changes in the basis vectors *du*, *dv*.

6 October 13

6.1 Review of 1st Fundamental Form

Def.[Symmetric Bilinear Forms]
Given arbitrary "1-forms" $\alpha, \beta \in T_p^*\mathbb{R}^n$, we can compose them to yield a symmetric, bilinear form: $\alpha \cdot \beta$ that is given by:

$$\alpha \cdot \beta(\vec{X}, \vec{Y}) = \frac{1}{2}[\alpha(\vec{X}) \cdot \beta(\vec{Y}) + \alpha(\vec{Y}) \cdot \beta(\vec{X})]$$

such that $\vec{X}, \vec{Y} \in T_p\mathbb{R}^n$.

Ex.
$\alpha \cdot \beta(X, X) = \frac{1}{2}(\alpha(X)\beta(X) + \alpha(X)\beta(X)) = \alpha(X)\beta(X)$
Note: it's symmetric, so $\alpha \cdot \beta(X, Y) = \alpha \cdot \beta(Y, X)$

Ex.
In $\mathbb{R}^2$, define "1-forms": $\alpha = x_1 dx_1 - dx_2$, $\beta = x_1 x_2 dx_2$ and vectors in the

tangent space at point p: $X = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$. $Y = \frac{\partial}{\partial x_2}$.

Then $\alpha(X) = x_1 + x_2$, $\alpha(Y) = -1$ and $\beta(X) = -x_1 x_2^2$, $\beta(Y) = x_1 x_2$.
So

$$\begin{aligned}
\alpha \cdot \beta(X, Y) &= \frac{1}{2}[(x_1 + x_2)(x_1 x_2) + (-1)(-x_1 x_2^2)] \\
&= \frac{1}{2}[x_1^2 x_2 + x_1 x_2^2 + x_1 x_2^2] \\
&= \frac{1}{2}[x_1^2 x_2 + 2x_1 x_2^2]
\end{aligned}$$

Consider if we do the function composition first:

$$\alpha \cdot \beta = (x_1 dx_1 - dx_2) \cdot x_1 x_2 dx_2) = x_1^2 x_2 dx_1 dx_2 - x_1 x_2 (dx_2)^2$$

So the basis for the new form (i.e. the composition of α and β) is $dx_1^2 = dx_1 dx_1$, $dx_1 dx_2 = dx_2 dx_1$, $dx_2^2 = dx_2 dx_2$.

Ex.
The dot product $\langle \cdot\ ,\ \cdot \rangle \subseteq \mathbb{R}^n$ is a symmetric bilinear form.

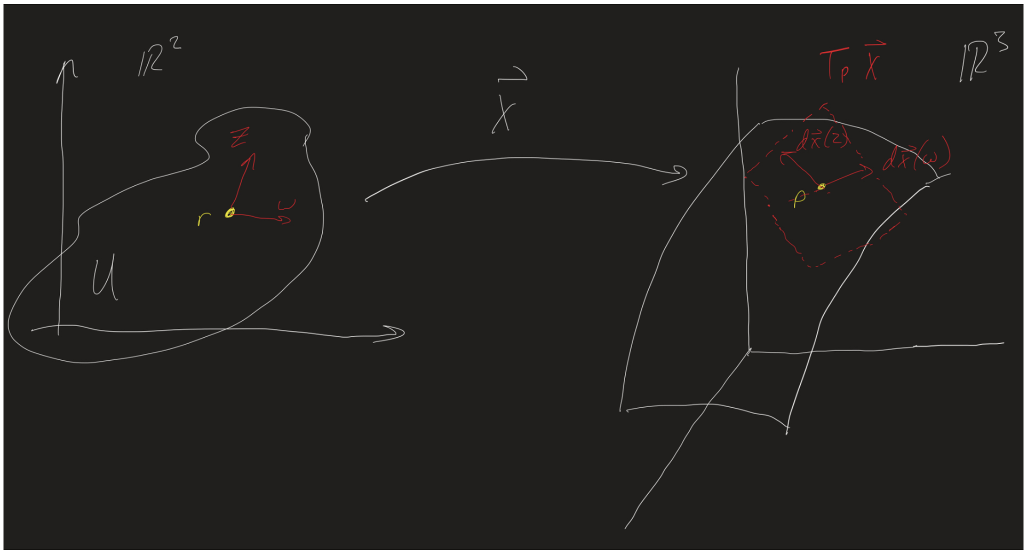
How is the dot product represented under our basis $dx_1^2, dx_1 dx_2, dx_2^2$?
$\langle \cdot\ ,\ \cdot \rangle = dx_1^2 + dx_2^2 + ... + dx_n^2$ is the new basis. Take two vectors in the tangent space:

$$\begin{aligned}
\langle \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\ ,\ \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \rangle &= \sum_{i=1}^{n} a_i b_i \\
&= (dx_1^2 + ... + dx_n^2)(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i})
\end{aligned}$$

Def. Dot product.
If α and β are vector-valued 1-forms [i.e. $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$, then $\alpha \cdot \beta(X, Y) = \frac{1}{2}(\alpha(X) \cdot \beta(Y) + \alpha(Y)\beta(X))$ where $\cdot$ is the dot product between two vectors.

Def. Differential.
The differential (derivative) of a surface x is a vectored-value "1-form": $\vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. Say that U is parameterized by (u, v), then $dx = x_u du + d_v dv$ [this should look intimately familiar].

Def.[First Fundamental Form]



$$\begin{tikzcd}
U_{z(t)} \arrow[r, "\vec{x}"] & \mathbb{R}^3 \\
I_t \arrow[u, "z"] \arrow[ur, "x(z(t))"'] &
\end{tikzcd}$$

So $\vec{x}$ maps (w,z) to $dx(w)$ and $dx(z)$ respectively which span the tangent space at point p [mapped from r in U $\rightarrow$ p in $\mathbb{R}^3$]. We will now show the first fundamental form corresponds to a composition of two vector-valued "1-forms":

$$\begin{aligned}
\mathcal{I}(w,z) &= dx(w) \cdot dx(v) \\
&= \frac{1}{2}[dx(w) \cdot dx(z) + dx(z) \cdot dx(w)] \\
&= \frac{1}{2}[(dx \cdot dx)(w,z)] \\
&\Rightarrow \mathcal{I} = dx \cdot dx \qquad\qquad = (x_u du + d_v dv) \cdot (x_u du + d_v dv)
\end{aligned}$$

Where $\vec{x}$ is our surface, and $d\vec{x}$ is the "differential" of this surface (and a vector-valued 1-form: $\begin{bmatrix} x_u du \\ x_v dv \\ dx \end{bmatrix}$.

$$\begin{aligned}
\mathcal{I} &= \vec{x}_u du^2 + 2\vec{x}_u \vec{x}_v du dv + \vec{x}^2 dv^2 \\
&= E du^2 + 2F du dv + G dv^2 \text{ where...} \\
E &= \vec{x}_u \cdot \vec{x}_u, F = \vec{x}_u \cdot \vec{x}_v, G = \vec{x}_v \cdot \vec{x}_v \\
&= [du dv] \begin{bmatrix} E, F \\ F, G \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}
\end{aligned}$$

6.2 Second Fundamental Form

The second fundamental form captures different "prototypical" information about the surface; most notably, that which relates to quadratic terms in the taylor series expansion about a point. We can reparameterize any smooth surface so that at any point p, the first-order terms $\to 0$ by aligning this point with the origin so that the plane z=0 is tangent to the surface at the origin (i.e. $f_x \to 0$, $f_y \to 0$).

The second fundamental form – like the first – is defined up to a sign:
Second fundamental form [caputes different features of the surface]. First fundamnetal form describes surface up to the 0th order. Second fundamental form describes the surface up to the 1st order [captures higher derivative information on the surface].

$$\begin{aligned}
\mathbb{I} &= -d\vec{x} \cdot d\vec{n} \\
&= -(\vec{x}_u du + \vec{x}_v dv) \cdot (\vec{n}_u du + \vec{n}_v dv) \\
&= -(\vec{x}_u \cdot v n_u du^2 + (\vec{x}_v \vec{n}_u + \vec{x}_u \vec{n}_v) du dv + \vec{x}_v \cdot \vec{n}_v dv^2) \\
&= -\vec{x}_u \cdot \vec{n}_u du^2 - (\vec{x}_v \vec{n}_u + \vec{x}_u \vec{n}_v) du dv - \vec{x}_v \cdot \vec{n}_v dv^2) \\
&\text{Let } e = -\vec{x}_u \cdot \vec{n}_u \text{ and } 2m = -(\vec{x}_v \vec{n}_u + \vec{x}_u \vec{n}_v) \text{ and } n = -\vec{x}_v \cdot \vec{n}_v \\
&= \begin{bmatrix} du & dv \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}
\end{aligned}$$

Where $e = -\vec{x}_u \cdot \vec{x}_u = \vec{x}_{u,u} \cdot n$;
$m = -\vec{x}_v \cdot \vec{n}_u = -\vec{x}_u \cdot \vec{n}_v = \vec{x}_{u,v} \cdot \vec{n}$ where the $\vec{x}_{u,u}$ is the second derivative of $\vec{x}$;
and $n = -\vec{x}_v \cdot \vec{n}_v = \vec{x}_{vv} \cdot \vec{n}$.

Note. $\vec{x}_v \cdot n = 0$. Can now take the derivative in the direction of u of both sides:

$$v x_{uv} \cdot \vec{n} + \vec{x}_v \cdot \vec{n}_u = 0$$

Summary The value of the first fundamental form:

$$I(a\vec{x}_u + b\vec{x}_v, c\vec{x}_u + d\vec{x}_v) = (a, b) \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

And the value of the second fundamental form:

$$\mathbb{I}(a\vec{x}_u + b\vec{x}_v, c\vec{x}_u + d\vec{x}_v) = (a, b) \begin{bmatrix} e & m \\ m & n \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

6.3 Computing Length of a Curve

We will now see how the first fundamental form might help us compute the length of a curve on a surface. Keep in mind that a curve on a surface is a 1-dimensional slice of that surface.

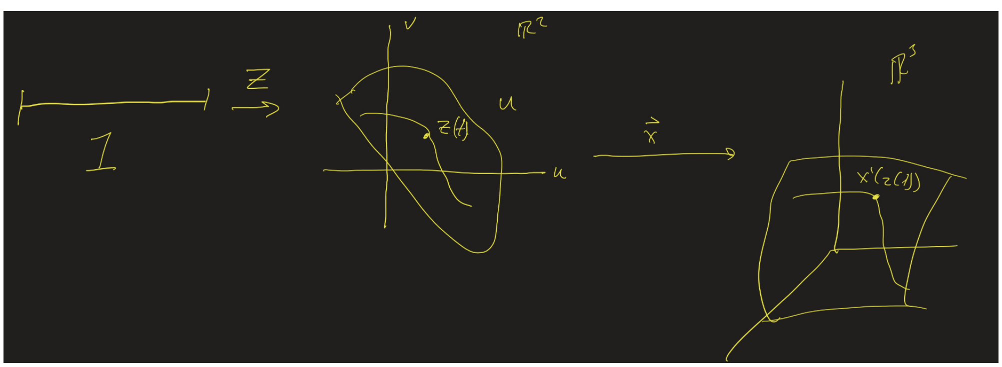
Let $\vec{x}$ be a surface: $\vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ and z a parameterized 2D curve: $z : \mathcal{I} \to U \subseteq \mathbb{R}^2$. Then the curve on the surface $\vec{x}$ is $\vec{x}(z(t))$.

Recall the velocity vector of the surface is the tangent vector: $\frac{d}{dt}(x(z(t))) = dx(z'(t))$ = how $\vec{x}$ changes with respect to small changes in $z(t)$ (i.e. $z'(t)$).

Theorem 6.1 *The arclength of $\vec{x}(z(t))$ between $t = a$ and $t = b$ is given by $\int_a^b \sqrt{I(z'(t), z'(t)}dt$.*

Proof:
Let $\gamma(t)$ be a curve in $\mathbb{R}^3$. Take $\gamma = \vec{x}(z(t))$ (this special curve).
We've seen previously the arclength is the integral of the speed along the curve:

$$
\begin{aligned}
arclength &= \int_a^b ||\gamma(t)||dt \\
&= \int_a^b ||\frac{d}{dt}\vec{x}(z(t))||dt \text{ expanding the definition of } |\gamma(t)| \\
&= \int_a^b ||d\vec{x}(z'(t))||dt \\
&= \int_a^b \sqrt{d\vec{x}(z'(t)) \cdot d\vec{x}(z'(t))}dt \\
&= \int_a^b \sqrt{I(z'(t), z'(t)}dt
\end{aligned}
$$



We now look at the relation between the acceleration of a curve and the second fundamental form.
Let $\vec{x}(z(t))$ be a curve on a surface $\vec{x}$. Then:

Theorem 6.2 $\vec{n} \cdot \frac{d^2}{dt^2}(\vec{x}(z(t)) = \mathbb{I}(z'(t), z'(t))$.

More informally, the projection of the acceleration of the curve onto the unit normal vector is equal to the second fundamental form. In short, the second fundamental form is how the acceleration changes in the normal direction (i.e. what component of the acceleration is in the normal direction). *Proof:*
The dot product between the tangent vector to the curve $\frac{d}{dt}(\vec{x}(z(t)))$ and normal vector to the curve $\vec{n}$ is 0:

$$0 = \frac{d}{dt}(\vec{x}(z(t))) \cdot \vec{n}$$

differentiating both sides.. + chain rule $\rightarrow$ product rule.

$$= \frac{d^2}{dt^2}(\vec{x}(z(t) \cdot \vec{n} + \frac{d}{dt}\vec{x}(z(t)) \cdot \frac{d}{dt}\vec{n}(z(t))$$

$$= \frac{d^2}{dt^2}\vec{x}(z(t)) \cdot \vec{n} + d\vec{x}(z'(t)) \cdot d\vec{n}(z'(t))$$

$$d\vec{x}(z'(t)) \cdot d\vec{n}(z'(t)) = 0 \qquad = \frac{d^2}{dt^2}\vec{x}(z(t)) \cdot \vec{n}$$

$$= \mathbb{I}(z'(t), z'(t))$$

Ex. [Sphere of Radius a]

Parametric Eqn: $\vec{x}(\theta, \phi) = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$,

$d\vec{x} = \vec{x}_\theta d\theta + \vec{x}_\phi d\phi = a \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix} d\theta + a \begin{bmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{bmatrix} d\phi$

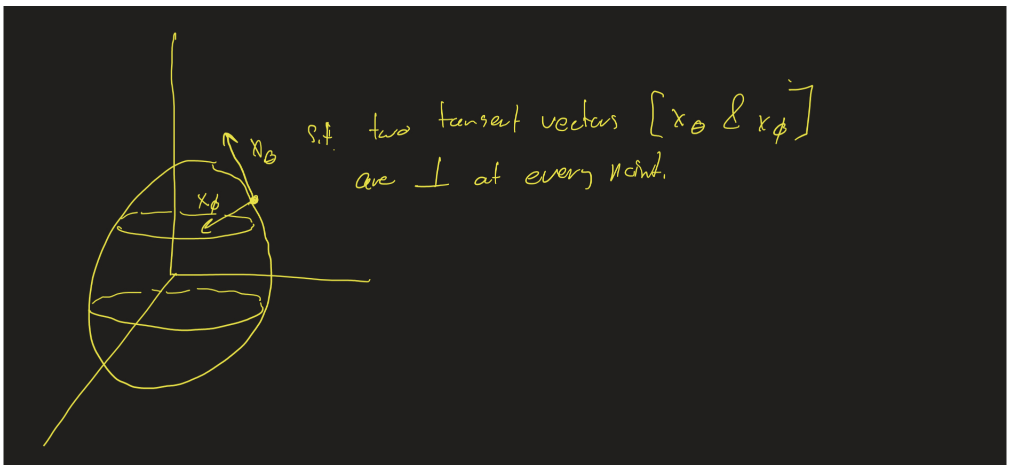
The first fundamental form:

$$\begin{aligned} \mathcal{I} &= d\vec{x} \cdot d\vec{x} \\ &= a^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

θ, ϕ are orthogonal coordinates from our spherical coord system (i.e. tangent vectors are orthogonal):
$\vec{x}_\theta \cdot \vec{x}_\phi = 0$

We now consider the second fundamental form in this context.
Note: The unit normal is simply the rescaled gradient of $\vec{x}$: $\vec{n} = \frac{1}{a}\vec{x}$. Then

$$
\begin{aligned}
\mathbb{I} &= -d\vec{x}d\vec{n} \\
&= -d\vec{x}(\frac{1}{a}d\vec{x}) \\
&= \frac{-1}{a}d\vec{x}\cdot d\vec{x} \\
&= \frac{-1}{a}\cdot \mathcal{I} \\
&= -a(d\theta^2 + sin^2\theta d\phi^2)
\end{aligned}
$$



Ex. [Surface of Revolution]

Let vector-valued 1-form: $\vec{x} = \begin{bmatrix} f(u)\cos\phi \\ f(u)\sin\phi \\ u \end{bmatrix}$

$d\vec{x} = \vec{x}_u du + \vec{x}_\phi d\phi = \begin{bmatrix} f'(u)\cos\phi \\ f'(u)\sin\phi \\ 1 \end{bmatrix} du + \begin{bmatrix} -f(u)\sin\phi \\ f(u)\cos\phi \\ 0 \end{bmatrix} d\phi.$

The first fundamental form is then given by:

$\mathcal{I} = d\vec{x}\cdot d\vec{x} = (f'^2(u)+1)du^2 + f^2(u)d\phi^2$

Note: It's significantly more difficult to compute $\vec{n}$: no longer in the same direction as the gradient (as it is on a sphere..):

$$\begin{aligned}
\vec{n} &= \frac{\vec{x}_u x \vec{x}_v}{||\vec{x}_u x \vec{x}_u||} \\
&= \frac{1}{\sqrt{f'(u)^2+1}} \begin{bmatrix} -\cos\phi \\ -\sin\phi \\ f'(u) \end{bmatrix} \\
\textbf{Note: } & \begin{bmatrix} -\cos\phi \\ -\sin\phi \\ f'(u) \end{bmatrix} \text{ is a normal vector } \perp \vec{x}_u, \vec{x}_u
\end{aligned}$$

From this $\vec{n}$, we can compute the differential:
$d\vec{n} = \vec{n}_u du + \vec{n}_\phi d\phi = \frac{1}{\sqrt{f'(u)^2+1}}(\begin{bmatrix} 0 \\ 0 \\ f''(u) \end{bmatrix} du + \begin{bmatrix} \sin\phi \\ -\cos\phi 0 \end{bmatrix} d\phi)$
Now we can finally compute the Second Fundamental Form:

$$\begin{aligned}
\mathbb{I} &= -d\vec{x} \cdot d\vec{n} \\
\mathbb{I} &= \frac{1}{\sqrt{f'^2(u)+1}}(-f''(u)du^2 + f(u)d\phi)
\end{aligned}$$

6.4 Principle Curvature

$\mathcal{I}, \mathbb{I}$ are symmetrix and bilinear.

First condition: $\mathcal{I} = I$ Using symmetric matrix theory of linear algebra, we can find two basis vectors $e_1, e_2 \in \mathbb{R}^2$ such that the first fundamental form under these two vectors is the identity matrix:
$\mathcal{I}(e_1, e_1) = 1, \mathcal{I}(e_2, e_2) = 1, \mathcal{I}(e_1, e_2) = \mathcal{I}(e_2, e_1) = 0$.
Equivalently stated, $\mathcal{I}$ under the basis $[e_1, e_2]$ is the identity matrix.

Second condition: $\mathbb{I}(e_1, e_2) = 0$. Equivalently stated, the matrix $\mathbb{I}$ under the basis $[e_1, e_2]$ is $\begin{bmatrix} k_1, 0 \\ 0, k_2 \end{bmatrix}$.
Where $k_1 = \mathbb{I}(e_1, e_1)$, $k_2 = \mathbb{I}(e_2, e_2)$.

The values $[k_1, k_2]$ are called the **principle curvatures** of the surface.

As we can always diagonalize a symmetric matrix, we can find such an $[e_1, e_2]$ so that condition 1 and condition 2 are satisfied.

Note: If $k_1 = k_2$ at a point p on the surface, then p is called the umbilic point. The sphere is the only known surface where all the points on the surface are umbilic points.

7 October 18

7.1 Umbilic point

Theorem 7.1 *Suppose $\vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is covered by umbilic points. Then $\vec{x}$ must be part of a sphere or plane.*

Umbilic points: $\mathbb{I} = k * \mathcal{I}$: where k is some function: might be different at different points.
Let the surface be parameterized by (u,v), and $\vec{n}$ is the normal vector.
$\vec{n}_v = -k\vec{x}_v$ and $\vec{n}_u = -k\vec{x}_u$.
So $\vec{n}_{vu} = -k_u\vec{x}_v - k\vec{x}_{uv}$ [differentiating w.r.t. u]
and $\vec{n}_{uv} = -k_v\vec{x}_u - k\vec{x}_{uv} \Rightarrow k_u\vec{x}_v = k_v\vec{x}_u$ and $\vec{x}_u, \vec{x}_v$ are basis vectors of the tangent space. So this can only occur when $k_u = 0 = k_v \Rightarrow k$ is a constant. **Case 1**: $k = 0$, so $\vec{n}_u = \vec{n}_v = 0$. Therefore, $d\vec{n} = 0 \to \vec{x}$ is part of a sphere [or a plane]. **Case 2**: $k \neq 0$ [but it must equal a constant]. To show $\vec{x}$ is a sphere, we must have $c = \vec{x} + \frac{1}{k}\vec{n}$. Therefore, taking the partial w.r.t. u: $c_u = \vec{x}_u + \frac{1}{k}\vec{n}_u = 0$ and taking the partial w.r.t. v: $c_v = \vec{x}_v + \frac{1}{k}\vec{n}_v = 0$. So c is a constant vector:
$\Rightarrow$ c is a constant vector:
$|\vec{x} - c| = |\frac{1}{k}\vec{n}| = \frac{1}{|k|}$.
$\Rightarrow \vec{x}$ is part of a sphere (centered at C or radius $\frac{1}{|k|}$.

7.2 Shape Operator

Shape operator: S = operator of the second fundamental form [defines a symmetric linear transformation].
$S : T_p\vec{x} \to T_p\vec{x}$. and $w \to -D_w\vec{n}$ [take the directional derivative of the unit normal in the direction of w: variation of the unit normal in the direction of w].
WTS: S is symmetric:
$< Sw, z > = < w, Sz >$ for $w, z \in T_P\vec{x}$.

S has two eigenvalues because it's symmetric: k_1, k_2 which relates to the principle curvature we defined last time.

with orthonormal eigenvectors x_1, x_2 [called principle directions].

Associate a bilinear form to S:
$w, z \in T_p\vec{x} \to < Sw, z >$, then
$\mathbb{I} = -\begin{bmatrix} \vec{n}_u \cdot \vec{x}_u & \vec{n}_u \cdot \vec{x}_v \\ -vn_v \cdot \vec{x}_u & \vec{n}_v \cdot \vec{x}_v \end{bmatrix}$ (under basis $\vec{x}_u, \vec{x}_v$).
$\mathbb{I} = -\begin{bmatrix} < S(\vec{x}_u), \vec{x}_u > & < S(\vec{x}_u), \vec{x}_v > \\ < S(\vec{x}_v), \vec{x}_u > & < S(\vec{x}_v), \vec{x}_v > \end{bmatrix}$
Where $< S(\vec{x}_u), \vec{x}_u > = < -\vec{n}_u, \vec{x}_u > = -\vec{n}_u \cdot \vec{x}_u$.

How to compute k_1, k_2 from $\mathbb{I}$?
Step 1 Solve $\det(\mathbb{I} - \lambda\mathcal{I}) = 0 \Rightarrow \lambda = k_1, k_2$. k_1, k_2 are the principle curvatures.

Recall $\mathcal{I} = \begin{bmatrix} \vec{x}_u\vec{x}_u & \vec{x}_u\vec{x}_v \\ \vec{x}_v\vec{x}_u & \vec{x}_v\vec{x}_v \end{bmatrix}$.

Step 2 Solve $(\mathbb{I} - \lambda\mathcal{I}) \cdot X = 0 \Rightarrow X_1, X_2$. X_1, X_2 are the principle directions.

Often k_1, k_2 are very difficult to compute, so instead we might compute the:
Gauss Curvature: $k = k_1k_2$.
Mean Curvature: $H = \frac{k_1+k_2}{2}$.

Special case: if we have a **curvature coordinate** (u,v): both $\mathcal{I}$ and $\mathbb{I}$ are diagonal under the basis $\vec{x}_u, \vec{x}_v$. Then $\vec{x}_u, \vec{x}_v$ are the principle directions and orthogonal to each other [Symmetric matrix: two eigenvectors are orthogonal].
In this special case...
$\mathcal{I} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\mathbb{I} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$.

Goal:
Find out k_1, k_2: $det(\mathbb{I} - \lambda\mathcal{I}) = det(\begin{bmatrix} \alpha - \lambda a & 0 \\ 0 & \beta - \lambda b \end{bmatrix})$

$= (\alpha - \lambda a)(\beta - -\lambda b)$
$\lambda = \frac{\alpha}{a}, \frac{\beta}{b}$
So $x_1 = \vec{x}_u$, $x_2 = \vec{x}_v$
So $\vec{x}_u$ and $\vec{x}_v$ are k_1, k_2.

Ex. Sphere of radius a: $\mathcal{I} = a^2 \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 u \end{bmatrix}$ $\mathbb{I} = \frac{-1}{a}\mathcal{I} = -a2 \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 u \end{bmatrix}$.
Since the matrix is symmetric, we can use the special case: $k_1 = \frac{-1}{a}, k_2 = \frac{-1}{a}$, and we see $k_1 = k_2 \Rightarrow$ umbilic points.

Expanding on the computations... $\vec{n} = \frac{1}{a}\vec{x}$ [centered at 0]
$d\vec{n} = \frac{1}{a}d\vec{x}$
$\mathbb{I} = -d\vec{n} \cdot d\vec{x} = \frac{-1}{a}d\vec{x} \cdot d\vec{x} = \frac{-1}{a}\mathcal{I}$ [$d\vec{x}d\vec{x}$ is the first fundamental form.]
$\Rightarrow \mathbb{I} = \begin{bmatrix} -\vec{n}_u \cdot \vec{x}_u & - \\ - & - \end{bmatrix}$

Ex. Surface of Revolution
$x = f(z)$ and rotate around the z-axis. $\begin{bmatrix} f(u)\cos v \\ f(u)\sin v \\ u \end{bmatrix}$, $f(u) > 0$.

$\vec{x}_u = \begin{bmatrix} f'(u)\cos v \\ f'(u)\sin v \\ 1 \end{bmatrix}$.

$\vec{x}_v = \begin{bmatrix} -f(u)\sin v \\ f(u)\cos v \\ 0 \end{bmatrix}$.

$\vec{n} = \frac{1}{\sqrt{1+(f'(u))^2}} \begin{bmatrix} -\cos v \\ -\sin v \\ f'(u) \end{bmatrix}$.

Can compute $\mathcal{I}, \mathbb{I}$ [and both are diagonal matrices], we can compute k_1, k_2 with ease: $k_1 = \frac{-f''(u)}{(1+(f'(u))^2)^{3/2}}$, $k_2 = \frac{\beta}{b} = \frac{1}{f(u)(1+f'(u)^2)^{1/2}}$ with $X_1 = \vec{x}_u$ and $X_2 = \vec{x}_v$.

Recall a planar curve, $x = f(z)$, we can compute the curvature of z, and the curvature of this curve is exactly equal to k_1 (or using u as a parameterization): $\begin{bmatrix} u \\ f(u) \end{bmatrix}$.

Ex. Graph of a function [also defines a surface]
Graph: z = f(x,y)
Surface $\vec{x} = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$
$\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ f_u \end{bmatrix}$, $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ f_v \end{bmatrix}$, $\vec{n} = \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{bmatrix} -f_u \\ -f_v \\ 1 \end{bmatrix}$
$x_u x_u = 1 + f_u^2$
$x_u x_v = f_u f_v$
$x_v x_v = 1 + f_v^2$
$\mathcal{I} = \begin{bmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{bmatrix}$
$\mathbb{I} = \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{bmatrix} f_{uu} & f_{uv} \\ f_u v & f_v v \end{bmatrix}$
Goal We want to simplify $\mathcal{I}, \mathbb{I}$ by choosing different coordinates.
Claim: We can assume $f_u = f_v = 0$ and $f_{uv} = 0$.
Can accomplish $f_u = f_v = 0$ via a rigid motion transformation.

Theorem 7.2 *For any surface $\vec{x}$, we can choose (u,v) [which are coordinates] such that $\vec{x}(u,v)$ is a graph $= \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$ and $\begin{cases} f_u = f_v = 0 \\ f_{uv} = 0 \end{cases}$ at u = 0, v= 0.*

So under the special coordinates:
$\mathcal{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\mathbb{I} = \begin{bmatrix} f_{uu} & 0 \\ 0 & f_{vv} \end{bmatrix}$
$\Rightarrow k_1 = f_{uu}$, $x_1 = \vec{x}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\Rightarrow k_2 = f_{vv}$, $x_2 = \vec{x}_v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
Taylor expansion of $f(u,v) = f(0,0) + f_u(0,0)u + f_v(0,0)v + \frac{1}{2} f_{uu}(0,0)u^2 +$

$f_{uv}(0,0)uv + \frac{1}{2}f_{vv}(0,0)v^2 + ...$ is the expansion at 0.

Then the first order and f_{uv} zero out so $f(u,v) = frac12k_1u^2 + frac12k_2v^2 + ...$

Very nice expression of the function that reviews the relationship between the surface and principle curvatures. The surface is given by f(u,v), but the surface is locally determined by k_1, k_2, so we know what the surface looks like at small neighborhoods around that point. **Taylor series expansion only legit when u and v are small $\Rightarrow$ holds true for local geometry**. **Conclusion:** The principle curvature k_1, k_2 "determine" the surface in a small neighborhood of p.

If we have a point, we can find a parameterization such that we move it to the origin, and then we have how the function looks like locally.

For a planar curve, can always realize a curve as a graph of a function: y = f(x) $\rightarrow \begin{bmatrix} u \\ f(u) \end{bmatrix}$ where $f(u) = \frac{1}{2}ku^2 + higherorderterms$ where k is the curvature at 0.

Next time: normal curvature.

7.3 Mean and Gauss Curvature

7.4 Normal Curvature, Euler Theorem

8 October 20

8.1 Local Surfaces

For any surface $\vec{x}$, we can locally parameterize it by a graph [as long as it is smooth]:

$\vec{x}(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$

Taylor expansion [0 and first order terms vanish b/c we choose fn. to reside at origin] $= \frac{1}{2}k_1u^2 + \frac{1}{2}k_2v^2 +$ higher order terms.

$f_{uu} = k_1$, $f_{vv} = k_2$, $f_{uv} = 0$.

The geometry around a base point is determined by these two curvatures (k_1, k_2).

True in a neighborhood around (0,0) [taylor expansion].

Case 1 We say p is elliptic if $k_1k_2 > 0$. [Gauss Curvature is positive].

Case 2: We say p is hyperbolic if $k_1K_2 < 0$ (Gauss Curvature is negative].

e.g. $k_1 = 1, k_2 = -1$ $f(u,v) = \frac{1}{2}u^2 - \frac{1}{2}v^2$. Hyperbolic paraboloid [saddle point at (0,0)]. **Case 3:** We say p is a parabolic point if $k_1k_2 = 0$ but $k_1 + k_2 \neq 0$ [equivalently Gauss Curvature = 0, but mean curvature $\neq$ 0]. Exactly one of $k_1, k_2 = 0$, but not at the same time.

e.g. $k_1 = 1, k_2 = 0$, $f(u,v) = \frac{1}{2}u^2$. Therefore, every surface **must** locally look like one of these three models.

8.2 Normal Curvature

Given a unit vector $v \in T_p\vec{x}$ [tangent space of the surface $\vec{x}$]. Consider the normal slice of $\vec{x}$ given by span$(\vec{n}, v) \cap \vec{x}$. Plane spanned by the normal vector and the tangent plane produces a curve $(\gamma(s))$ passing through p, then $\gamma'(0) = p$, $\gamma'(0) = v$.

So using the notations from the curve's section, [in particular, planar curves]:
$T = v$ at p, $T' = kN$ where k is the curvature at p. Since we have that N lies in the plane spanned by $span(\vec{n}, v)$, and $N \perp T$, N = $\pm\vec{n}$.

$\pm k = T' \cdot \vec{n}$ [take dot product on both with $\vec{n}$ goes to $\vec{n} \cdot \vec{n} = \pm 1$ b/c unit vector.]
$\pm k = T' \cdot' vn = -T \cdot \vec{n}_v$take derivate w.r.t. v and $T \cdot \vec{n} = 0 = v \cdot -D_v\vec{n} = S(v) \cdot v =< S(v), v >$ S is the shape operator $= \mathbb{I}(v, v)$. where k is the normal curvature in the direction of v [up to a sign].

$S(v) \cdot v =< S(v), v >$. Can compute the shape operator of a tangent vector. Geometric meaning is the normal slice's curvature: the intersection of the normal plane with the surface.
However, this varies based on the tangent vectors chosen. So
Ex. Choose $v = X_1, X_2$ (principal directions), then the normal curvature is k_1, k_2.
$S(X_1) = k_1X_1$ [X_1 is the eigenvector].
$< S(X_1), X_1 >= k_1 < X_1, X_1 >= k_1$ [X is an orthonormal eigenvector].

8.3 Euler Formula

What is the relation of a general normal curvature to the two principle curvatures. This is given by Euler's formula.
$v = \cos\theta X_1 + \sin\theta X_2$ [X_1, X_2 are the orthonormal basis, and v is a unit vector, so it can be written as a linear combination of sin and cos the orthonormal basis].
$< S(v), v >=< S(\cos\theta \cdot X_1 + \sin\theta X_2), cos\theta X_1 + \sin\theta X_2 >$
$= < \cos\theta \cdot k_1X_1 + \sin\theta k_2X_2, \cos\theta X_1 + \sin\theta X_2 >$
$= < k_1 \cdot cos^2\theta + k_2 \sin^2\theta >\Rightarrow$ normal curvature always lies between k_1 and k_2.
Normal curvature is always between k_1 and k_2.

If $k_1k_2 < 0$ (hyperbolic point), then can find $v = \cos\theta X_1 + \sin\theta X_2$ so that $< S(v), v >= 0$.
Call this v to be the asympotic direction [direction in which the normal curvature vanishes].

Ex. Choose $\vec{x} = \begin{bmatrix} u \\ v \\ \frac{1}{2}u^2 - v^2 \end{bmatrix}$ is a surface parameterized by a graph, $k_1 = 1$, $k_2 = -1$
$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}_u$, $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{x}_v$. Then Choose $\theta = \frac{\pi}{4} \to v = \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2$
Then $k_1 cos^2\theta + k_2 \sin^2\theta = 1 * (cos(\frac{\pi}{4})^2) - 1 * \sin(\frac{\pi}{4})^2 = 0$. Asymptotic direction is $\theta = \frac{\pi}{4}$. $y = x$ is the asymptotic lines. [may be $\pm$ for both theta and the line]

Ex. Sphere: elliptic points.
$k_1 = k_2 \neq 0$

Ex. Cylinder: parabolic points. Two
As you change the vector from $x_2 \to x_1$, the normal vector will become an ellipse.
Ex. Torus in $\mathbb{R}^3$

8.4 Exterior Calculus

Recall 1-form α at a point $p \in \mathbb{R}^n$, $\alpha : T_p\mathbb{R}^n \to \mathbb{R}$,
$\alpha = \alpha_1 dx_1 + ... + \alpha_n dx_n$ [written in terms of the basis dx_i: dual basis of basis in the tangent space in the tangent space: $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ such that $dxi = \frac{\partial}{\partial x_j} = \delta_{ij}$.
Differential 2-form $dx_i \vee dx_j$ that satisfies the following rules:

- $dxi \vee dxj = -dx_j \vee dxi$ [anti-symmetric]
- $dx_i \vee dx_i = 0$

A general 2-form is $\sum_{i<j} \alpha_i j dx_i \vee dx_j$
Ex. n = 2, $\mathbb{R}^n = \mathbb{R}^2$
$dx_1 \vee dx_2$ is a 2-form.
$(x_1 x_2) * dx_1 \vee dx_2$ [coefficients change based on x_1, x_2] is a 2-form. $f(x_1, x_2) \cdot dx_1 \vee dx_2$ is the only 2-form in $\mathbb{R}^2$. Basis are dx_1, dx_2 are the basis for the 1-form can be used to form the basis for the two forms: $dx_1 \vee dx_2, dx_1 \vee dx_2 dx_2 \vee dx_1, dx_2 \vee dx_2 = 0, -1 * theother, -1 * theother, 0$, so we can write the basis in the form $dx_1 \vee dx_2$.
When $n = 3$, a 2-form is:
$\alpha = \alpha_{12} dx_1 \vee dx_2 + \alpha_{13} dx_1 \vee dx_3 + \alpha_{23} dx_2 \vee dx_3$, and $dx_1 \vee dx_2, dx_2 \vee dx_3, dx_1 \vee dx_3$ are the basis of the space of 2-forms.
2-form is a bilinear function, $\alpha : T_p\mathbb{R}^n x T_p\mathbb{R}^n \to \mathbb{R}$, so $(w, z) \to \alpha(w, z)$. More generally, we can define k-forms as linear function on $T_p\mathbb{R}^n$ on the tangent space, $\alpha : T_p\mathbb{R}^n x ... x T_p\mathbb{R}^n \to \mathbb{R}$ [cross product of k many tangent spaces].
$(w_1, ..., w_k) \to \alpha(w_1, ..., w_k)$.
Bilinear functions must be anti-symmetric [skew symmetric – interchanging any two vectors changes the sign]: $\alpha(w, z) = -\alpha(z, w)$. $\alpha(w_1, ..., w_i, ...w_j, ...w_k) = -\alpha(w_1, ..., w_j, ..., w_i, ...w_k)$ where we interchange i and j $(i \neq j)$, then the sign

changes.

Ex. $dx_1 \vee dx_2(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix})$

Ex $dx \vee xy \vee dz(v_1, v_2, v_3) \in \mathbb{R}^3$

$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix}$

$v_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix}$ $v_3 = \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix}$ $=det(\begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{bmatrix}$

Geometric meaning: volume of the polygon spanned by v_1, v_2, v_3

Exterior Derivative $\vee^k \mathbb{R}^n = \{\text{all k-forms in } \mathbb{R}^n\}$
$d : \vee^k \mathbb{R}^n \to \vee^{k+1} \mathbb{R}^n$
In particular when $k = 0$, $\vee^0 \mathbb{R}^n = \{\text{functions on } \mathbb{R}^n\}$
$d : \vee^0 \mathbb{R}^n \to \vee^1 \mathbb{R}$

If we have a $k - form$, $\alpha = adx_{i1}..^d x_{ik}$ where a is a function in $\vee^0 \mathbb{R}^n$, then

9 October 20 (TA Review)

9.1 Bilinear Form

Let V be a vector space over $\mathbb{R}$. A bilinear form on V is a bilinear function: f: $VXV \to \mathbb{R}$ where $V = \mathbb{R}^n$ and $\phi(u, v) = \sum\limits_{i=1}^{n} u_i v_i$.

Ex.
$V = \mathbb{R}^n$, $A \in \mathbb{R}^{nxn}$, $\phi(u, v) = u^T Av = \sum_{i,j=1}^n u_i A_{ij} v_j$

Def[Symmetric Bilinear Form]: A symmetric blilinear form ϕ which satisfies $\phi(x, y) = \phi(y, x)$ $\forall x, y$.

Ex. $\langle \cdot\, , \,\cdot \rangle$ is symmetric.

Proposition 9.0.1 *Every bilinear form on $\mathbb{R}^n$ is symmetric.*

Proof: Let ϕ be a bilinear form. Let $A_{ij} = \phi(e_i, e_j)$. Then $\phi(u, v) = \phi(\sum v_i e_i, \sum v_j e_j) = \sum_{i,v} \phi(e_i, e_j) v_i v_j$ where $\phi(e_i, e_j) = A_{i,j}$ where $e_k = \begin{bmatrix} 0 \\ ... \\ 1 \\ 0 \\ ... \end{bmatrix}$

Proposition 9.0.2 *Let V be a n-dimensional vector space over the field $\mathbb{R}$ with basis $[v_1, .., v_n]$. Let ϕ be a bilinear form on V and let A be the matrix of ϕ with respect to $\{v_i\}$. Then ϕ symmetric $\iff$ A is symmetric.*

Proof: Suppose ϕ is symmetric. Then $A_{ij} = \phi(v_i, v_j) = \phi(v_j, v_i) = A_{ji}$.
Suppose A is symmetric. Then $\phi(x, y) = x^T Ay = (xTAy)^T = (y^T A^T x) = y^T Ax = \phi(y, x)$.
Note: this uses the following property of matrices: $(AB)^T = B^T A^T$.

Def: ϕ is non-degenerate if $\forall x \in V$, if $\forall y \in V$, $\phi(x, y) = 0$, then $x = 0$.

Proposition 9.0.3 *ϕ is non-degenerate $\iff$ A is non singular $\iff$ det(A) $\neq 0 \iff$ A is invertible.*

Def ϕ is an inner product $\iff$ ϕ is nondegenerate and symmetric.
Ex. The dot product.
Ex. A symmetric, A non singular, $\langle \cdot\ ,\ \cdot \rangle xy = x^T Ay$.

Proposition 9.0.4 *$\mathcal{I}$ is an inner product on T_pX.*

Let v_s, v_t be the basis of the space spanned by T_pX. Then $A = \begin{bmatrix} v_s * v_s & v_s v_t \\ v_s v_t & v_t v_t \end{bmatrix}$ is representative of I up the the basis v_s, v_t.

9.2 k-forms

Given a k-form α and an l-form β, $\alpha \vee \beta$ is a (k+l)-form.

Let α, β be 1-forms. i.e. $T_p \to \mathbb{R}$ where the transformation from the tangent space is linear.
Then $\alpha \vee \beta : T_p x T_p \to \mathbb{R}$ where the transformation is again, linear.

$(\alpha \vee \beta)(x, y) = \frac{1}{2}[\alpha(x)\beta(y) - \beta(x)\alpha(y)$

Def. A bilinear form on a vector space V over $\mathbb{R}$ is a function $\phi : V x V \to \mathbb{R}$ such that it is linear in each argument:
$\phi(ax_1 + bx_2, y) = a\phi(x_1, y) + b\phi(x_2, y)$.

Def. A bilinear form is alternating (or antisymmetric) if $\forall x, y\ \phi(x, y) = -\phi(x, y)$.
Ex. ϕ is alternating, then $\phi(x, x) = 0$.
Def. A 2-form of V is an alternating bilinear form [unlike the inner product which is a symmetric linear form]: $x^T Ay$ where switching the order of x and y will yield the opposite result. This is because $A = -A^T$.

Proposition 9.0.5 *Choose a basis $[v_1, v_2, ...v_n]$ for V. Then every 2-form is of the form: $(x, y) \to x^T Ay$ where is is an antisymmetric nxn matrix ($A^T + A = 0$).*

10 October 25

- moving frame of a map
- moving frame of a surface.
- moving frame of a surface.

Def.[Exterior Derivative]:
$d : \{k - forms\} \to \{(k+1) - forms\}$.
$\mathrm{d}(\alpha(x_1, ..., x_n)dx_i \vee ... \vee dx_{i_k}) = \mathrm{d}\alpha \vee dx_i \vee ... \vee dx_{i_k}$: linear combination of the basis vectors dx_{i_k} where α are the coefficients in front of the basis vectors.

Theorem 10.1 $d^2 = 0$

Proof: Only for 0-forms
$\{0 - formsin\mathbb{R}^n\} = \{smoothfunctionsin\mathbb{R}^n\}$
$\Rightarrow d^2 f = 0$

$$f = f(x_1, ..., x_n)$$

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Def.[Moving Frame]
$\vec{x} : U \subseteq \mathbb{R}^m \to \mathbb{R}^3$, $\vec{x}(u_1, ..., u_m) = \begin{bmatrix} x_1(u_1, ..., u_m) \\ x_2(u_1, ..., u_m) \\ x_3(u_1, ..., u_m) \end{bmatrix}$
A moving frame of $\vec{x}$ is a tripe of vectors: $e_i : U \to \mathbb{R}^3$, $\{e_1, e_2, e_3\}$ which is an [arbitrary] orthonormal basis (oriented) of $\mathbb{R}^3$ at each point.

Ex. $\vec{x} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ that maps $(\gamma, \phi, z) \to \begin{bmatrix} rcos\phi \\ r\sin\phi \\ z \end{bmatrix}$ where $\begin{bmatrix} \gamma > 0 \\ \phi \in [0, 2\pi) \\ z \in \mathbb{R} \end{bmatrix}$
[notice these are cylindrical polar coordinates].
Choose the following moving frame: $\epsilon_1 = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} = \vec{x}_r$,
$\epsilon_2 = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} = \gamma^{-1} * \vec{x}_\phi$
$\epsilon_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{x}_z$

Def. Exterior derivative of a map $\vec{x}$.

$$\vec{x} = \begin{bmatrix} x_1(u_1,...,u_m) \\ x_2(u_1,...,u_m) \\ x_3(u_1,...u_m) \end{bmatrix}, \; d\vec{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_i} du_i \\ \sum_{i=1}^{n} \frac{\partial x_2}{\partial u_i} du_i \\ \sum_{i=1}^{n} \frac{\partial x_3}{\partial u_i} du_i \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial x_1}{\partial u_i} \\ \frac{\partial x_2}{\partial u_i} \\ \frac{\partial x_3}{\partial u_i} \end{bmatrix} * du_i =$$

$$\sum_{i=1}^{n} \vec{x}_{u_i} * du_i$$

$d\vec{x}$ is a 3-vector valued 1-form.

Ex. Find out $d\vec{x} when \vec{x}$ is the polar cylindrical coordinate map: $d\vec{x} = \vec{x}_r dr + \vec{x}_\phi d\phi + \vec{x}_z dz) = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} dr + \begin{bmatrix} -r\sin\phi \\ r\cos\phi \\ 0 \end{bmatrix} d\phi + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dz.$

Also called the differential derivative of x.

Def $\theta_k = d\vec{x} e_k$ where $\{e_k\}_{k=1}^{3}$ is a moving frame of $\vec{x}$.
Want to study how the map $\vec{x}$ changes:

- How $\vec{x}$ changes along the moving frame
- How the moving frame changes along itself.

θ_k (scalar-valued) 1-form: captures (1).

Ex. In the polar coordinate map: $d\vec{x} = \vec{x}_r dr + \vec{x}_\phi d\phi + \vec{x}_z dz$, $\theta_1 = d\vec{x} \cdot e_1 = d\vec{x} \cdot \vec{x}_r = \vec{x}_r \vec{x}_r dr = dr$
$\theta_2 = d\vec{x} \cdot e_2 = d\vec{x} \cdot r^{-1} \vec{x}_\phi = \vec{x}_\phi r^- 1 \vec{x}_\phi d\phi = r d\phi$
$\theta_3 = d\vec{x} \cdot e_3 = d\vec{x} \cdot \vec{x}_z = \vec{x}_z \vec{x}_z dz = dz$
Therefore, $d\vec{x} = \sum_{k=1}^{3} \theta_k e_k = \theta_1 e_1 + \theta_2 \epsilon_2 + \theta_3 \epsilon_3$ [$d\vec{x}$ is a 3-vector-valued 1-form and we represent it with the coordinates that are the scalar-valued 1-forms θ_i with a different orthonormal basis dr becomes de_1].

This answers our first question; how $\vec{x}$ changes along our moving frame.
$d\vec{x} = E\theta$ where E is a 3x3 matrix of the orthonormal basis vectors: $\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$.
To study how a smooth map changes, we looked at its differential. Similarly to study how the moving frame changes, we will again examine its differential.

Def $w_{jk} = e_j de_k = de_k e_j$ where (w_{jk} are coordinates of de_k under the moving frame).
Want to represent a vector in $\mathbb{R}^3$ in the basis spanned by $\{e_1, e_2, e_3\}$. Example: $v \in \mathbb{R}^3$, $v_j = v \cdot e_j$ and $v = \sum_{j=1}^{3} v_j \cdot e_j \Rightarrow$ converting vectors from basis e_i to $\mathbb{R}^3$.

$de_k = \sum_{j=1}^{3} w_{jk} e_k$
Lemma $w_{jk} = -w_{kj}$
$W := (w_{jk})$ matrix is skew-symmetric.
Proof: $w_{jk} = e_j de_k = -de_j \cdot e_k = -w_{kj}$.

$$W = \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix}$$

Theorem 10.2 $dE = E \cdot W$

$d(e_1, e_2, e_2) = (de_1, de_2, de_3) = (e_1, e_2, e_3)W$
Proof: Just plug in $de_k = \sum_{j=1}^{3} w_{jk} e_j$ and use lemma W is a skew-symmetric matrix.
This answers how the moving frame changes along itself.

Ex. $\vec{x}(t)$ be a curve in $\mathbb{R}^3$, arclength parameterization with curvature $\kappa(t)$, torsion $\tau(t)$.

Choose the moving frame to be (T, N, B) $= (e_1, e_2, e_3)$.
$d\vec{x} = \vec{x}'(t)dt = T \cdot dt$
$\Rightarrow \theta_1 = d\vec{x} \cdot T = T \cdot T dt = dt$
$\theta_2 = 0 = \theta_3$
$\theta_2 = d\vec{x} \cdot N = T \cdot dt \cdot N = 0$ $\theta_3 = d\vec{x} \cdot B = 0$
(1) $d\vec{x} = E \cdot \theta \iff T \cdot dt = (T, N, B) \begin{bmatrix} dt \\ 0 \\ 0 \end{bmatrix}$ How the curve changes along the moving the frame.
How the curve changes along the moving frame. The changes of x is decided by the changes in the tangent vector. (2): study changes in the moving frame along itself: dE $= E \cdot W \iff$ the frenet equation.
$$W = \begin{bmatrix} 0 & -\kappa dt & 0 \\ \kappa dt & 0 & -\tau dt \\ 0 & \tau dt & 0 \end{bmatrix}$$
$w_{12} = e_1 \cdot de_2 = T \cdot dN = T \cdot N'(t)dt = -\kappa dt$ Together, they give us a full picture of the curve.

Theorem 10.3 (first structure equation) *$d\theta + w \vee \theta = 0$ [W is our skew-symmetric matrix and $\theta = [\theta_1, \theta_2, \theta_3]$]*

equivalent to $d \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + W \vee \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$

$$\iff \begin{cases} d\theta_1 + w_{12} \vee \theta_2 + w13 \vee \theta_3 = 0 \\ d\theta_2 - w_{12} \vee \theta_1 + w23 \vee \theta_3 = 0 \\ d\theta_3 - w_{13} \vee \theta_1 - w23 \vee \theta_2 = 0 \end{cases}$$

Proof:
$d\vec{x} = E \cdot \theta$
$0 = d^2\vec{x} = d(E \cdot \theta) = [ruleofexteriorderivative] = dE \vee \theta + Ed\theta$. Using $d^2 = 0$.
$= EW \vee \theta + Ed\theta = E(W \vee \theta + d\theta)$ [E is invertible]
Therefore $0 = W \vee \theta + d\theta$ Recall: $d(f(x_1, .., x_n)dx_1 \vee dx_2) = df \vee dx_1 \vee dx_2 + f(d(dx_1 \vee dx_2))$ where the second term is 0 $(dx_1 \vee dx_2)$.
More generally, $d(f \vee \theta) = df \vee \theta + fd\theta$.

Theorem 10.4 (Second Structure Equation) $dw + w \vee w = 0$

W is again our skew symmetric matrix. Every coordinate is a 1-form. Scalar product is replaced by wedge product (effectively multiplying 1-forms). *Proof:*
$dE = EW$
$0 = d(dE) = d(EW)$ [=0 on LHS because $d^2 = 0$]
$0 = dE \vee W + EdW$ [special case E is a 0-form]
$d(\alpha \vee \beta) = d\alpha \vee \beta + (-1)^k \alpha \vee d\beta$ [general formula: if α is a k-form].

$= EW \vee W + EdW$
$= E(W \vee W + dW)$ [Multiply by E^{-1} on both sides]
$0 = W \vee W + dW$

Theorem 10.5 *$U \subseteq \mathbb{R}^m$ be simply connected [any loop can shrink to a point in that domain].*

Given W satisfying the second structure equation $dW + W \vee W = 0$, then $W^T = -W$ and given $\{e_1, e_2, e_3\}$ at $P \in U$, then $\exists$ a unique $\{e_1, e_2, e_3\}$ that extends $\{e_1, e_2, e_3\}$ at p so that $dE = EW$. If W satisfies the second structure equation, then $dE = EW$ is always solvable.
If given $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ and $d\theta + W \vee \theta = 0$, and specify $\vec{x}(p$ then $\exists$ a unique $\vec{x}$ satisfying $d\vec{x} = E \cdot \theta$ [you can go backwards].

11 November 1st

Midterm: material up through adaptive frames.

11.1 Curvature and Isometry

Isometry: when two surfaces are identical or not.
Def. Global Isometry:
$\vec{x} : U \to \mathbb{R}^3$ and $\vec{y} : \tilde{U} \to \mathbb{R}^3$ are globally isometric if there exists an isometry θ of $\mathbb{R}^3$ so that $\vec{y} = \theta \circ \vec{x}$

$\theta : x \to Ax + b$ where A is an orthogonal matrix and b is some constant vector. $\vec{y} = A\vec{x} + b$: if we can find the matrix A and the column vector b, then they are isometric.

Theorem 11.1 *If two surfaces are globaly isometric, they have equal first and second fundamental form. If $\vec{y} = \theta\vec{x}$ for some isometry θ of $\mathbb{R}^3$, then $\mathcal{I}\vec{y} = \mathcal{I}\vec{x}$ and $\mathbb{I}\vec{y} = \mathbb{I}\vec{x}$*

Proof $\vec{y} = A\vec{x} + b$
$d\vec{y} = Ad\vec{x}$
$\mathcal{I}\vec{y} = d\vec{y} \cdot d\vec{y} = d\vec{y}^T \cdot d\vec{y} = d\vec{x}^T A^T A d\vec{x} = d\vec{x}^T d\vec{x} = \mathcal{I}\vec{x}$ using $A^T A = I$.
$\vec{n}_y = A \cdot \vec{n}_x$. Similarly $\mathbb{I}\vec{y} = \mathbb{I}\vec{x}$ (where $\mathbb{I} = -d\vec{n} \cdot d\vec{x}$).
θ is direction if $det(A) = 1$ [orientation preserving]. θ is indirect if $det(A) = -1$. Assuming A is a matrix of constants.

Every isometry of $\mathbb{R}^3$ is called rigid motion.

Question: Can we use only the $\mathcal{I}$ to compute κ (the gauss curvature)?
Last class: Step 1: find an adaptive frame e_1, e_2 [e_3 is the normal vector of the surface].
Step 2: Compute $d\vec{x} = e_1\theta_1 + e_2\theta_2$ to find out θ_1 and θ_2.
Step 3: Compute $w_{12} = e_1 de_2 = -e_2 de_1$
Step 4: Use Gauss equation to compute $dw_{12} = k\theta_1 \wedge \theta_2$.

Today: (1): Find $\mathcal{I} = d\vec{x} \cdot d\vec{x}$
(2): Write $\mathcal{I} = \theta_1^2 + \theta_2^2$ (by inspection). [in some cases, this is easier to do]. Write first fundamental form as a score sum for some 1-forms θ_1, θ_2
(3): Solve the first structure equation: $d\theta_1 = w_{12} \wedge \theta_2$ to find w_{12}
$d\theta_2 = w_{12} \wedge \theta_1$.
(4) Use the Gauss Equation $dw_{12} = k\theta_1 \wedge \theta_2$ to find κ.

Justification:
Goal: find adaptive frame e_1, e_2 such that θ_1, θ_2 are the 1-forms in this decomposition: $d\vec{x} = e_1\theta_1 + e_2\theta_2$.

First, θ_1, θ_2 are linearly independent:
Suppose not. Then WLOG $\theta_1 = a\theta_2$: Space of 1-forms are 2-dimension space. $\mathcal{I} = \theta_1^2 + \theta_2^2 = (1 + a^2)\theta_2^2$. Can find $v \in T_pU$ such that $v \neq 0$ and $\theta_2(v) = 0$ [θ_2 is a linear function and therefore must have a non-trivial kernel mapping to 0], $\mathcal{I}(v, v) = (1 + a^2)\theta_2^2(v) = 0$ which is a contradiction since $\mathcal{I}$ is a non-degenerate 2-form [equality of $\mathcal{I}$ only achieved when $v = 0$].

This implies θ_1 and θ_2 form a basis of 1-form space: T_p^*U.

So let $u_1, u_2 \in T_pU$ that are dual basis of θ_1, θ_2 (this means $\theta_i(u_j) - \delta_{ij}$).

$e_1 = d\vec{x}(u_1) \in T_{\vec{x}(p)}\vec{x}$
$e_2 = d\vec{x}(u_2) \in T_{\vec{x}(p)}\vec{x}$ So we can check $d\vec{x} = e_1\theta_1 + e_2\theta_2$ and e_1, e_2 are orthonormal.

Can now find $w_{12} = e_1 de_2$ implicitly using the structure equations.

Ex. Surface of revolution:
$\vec{x} = \begin{bmatrix} f(u)\cos\phi \\ f(u)\sin\phi \\ u \end{bmatrix}$.
(1) $\vec{x}_u = \begin{bmatrix} f'(u)\cos\phi \\ f'(u)\sin\phi \\ 1 \end{bmatrix}$ $\vec{x}_v = \begin{bmatrix} -f(u)\sin\phi \\ f(u)\cos\phi \\ 0 \end{bmatrix}$.
$\mathcal{I} = (1 + f'(u)^2)du^2 + f^2(u)d\phi^2$
$\mathcal{I} = \theta_1^2 + \theta_2^2$
So $\theta_1 = \sqrt{1 + f'(u)^2}du$
and $\theta_2 = f(u)d\phi$
(3) Solve for w_{12} using the two structure equations:
$d\theta_1 = -w_{12} \wedge \theta_2$
$d\theta_1 = d(\sqrt{1 + f(u)^2}du) = 0$
$0 \Rightarrow -w_{12} \wedge \theta_2 = 0$ $[w_{12} = a\theta_1 + b\theta_2]$ w_{12} can be represented as a linear combination fo the basis θ_1 and θ_2. Therefore, $a = 0$ so $w_{12} = b\theta_2$ because otherwise $a\theta_1 \wedge \theta_2$ will be non-zero.

$d\theta_2 = w_{12} \wedge \theta_1$
$d\theta_2 = d(f(u)d\phi) = f'(u)du \wedge d\phi$
$w_{12} \wedge \theta_1 = w_{12} \wedge (\sqrt{1 + f'(u)^2}du) \Rightarrow w_{12} = something * d\phi$
$something = \frac{-f'(u)}{\sqrt{1+f'(u)^2}} \Rightarrow w_{12} = -\frac{f'(u)}{\sqrt{1+f'(u)^2}}d\phi$
(4) $dw_{12} = -\frac{f''(u)}{\sqrt{1+f'(u)^2}}du \wedge d\phi$.
$dw_{12} = \kappa\theta_1 \wedge \theta_2$
$dw_{12} = -\frac{f''(u)}{f(u)(1+(f'(u)^2))^3}\theta_1 \wedge \theta_2$
$\Rightarrow \kappa = -\frac{f''(u)}{f(u)(1+(f'(u)^2))^3}$

Ex
$\vec{x} = \begin{bmatrix} x^2 \\ y \\ y^2 \end{bmatrix}$
$vx_v = \begin{bmatrix} 2x \\ 0 \\ 0 \end{bmatrix}$
$\vec{x}_y = \begin{bmatrix} 0 \\ 1 \\ 2y \end{bmatrix}$ $\mathcal{I} = 4x^2dx^2 + (1 + 4y^2)dy^2$
$\theta_1 = 2xdx$, $\theta_2 = \sqrt{1 + 4y^2}dy$

$d\theta_1 = 0 = d\theta_2$
$\begin{cases} 0 = -w_{12} \wedge \theta_1 \Rightarrow w_{12} = a\theta_1 \\ 0 = -w_{12} \wedge \theta_2 \Rightarrow w_{12} = b\theta_2 \end{cases}$
$\Rightarrow a = b = 0$ and $w_{12} = 0 \Rightarrow \kappa = 0$.

11.2 Gauss' Theorem Egregium

Gauss curvature is something intrinsic: only need first fundamental form. **Intrinsic:** Objects that are only related to $\mathcal{I}$.
Extrinsic: Objects that are also related to $\mathbb{I}$.
Gauss curvature is intrinsic.

Theorem 11.2 (Gauss's Theorem Egregium) *Two surfaces that are locally isometric ($\mathcal{I}_{\vec{x}} = \mathcal{I}_{\vec{y}}$ everywhere in definition domain U), then $\vec{x}, \vec{y}$ have the same Gauss curvature κ.*

Proof:
By the computation process of κ by using only $\mathcal{I}_{\vec{x}} (= \mathcal{I}_{\vec{y}}$ [when first fundamental forms are equal, can carry out the previous procedure to find κ where they will be equal because the first fundamental form are equal).

Theorem 11.3 *Two flat surfaces [means that $\kappa = 0$]are locally isometric.*

For two surfaces $\vec{x}$ and $\vec{y}$ if $\kappa\vec{x} = \kappa\vec{y} = 0$, then we deduce the two surfaces are locally isometric (kind of the opposite direction of Gauss's Theorem Egregium).
Ex.
$\vec{x} = \begin{bmatrix} z\cos\phi \\ z\sin\phi \\ z \end{bmatrix}$ [surface of revolution of a line z = a cone].
Want to compute it's Gauss curvature κ of the cone:
Recall for a revolutionary surface: $\kappa = \frac{f''(z)}{f(z)(1+f'(z)^2)^3} = 0$ $f(z) = z$ and $f''(z) = 0$.
Therefore Gauss curvature is 0, so the cone is locally isometric to the plane. Can be "unwrapped" to the plane in $\mathbb{R}^2$.
Question: Are these 3 surfaces [flat cylinder, flat cone, and plane in $\mathbb{R}^2$] globally isometric?
Of course not: $\mathbb{I}$ is different in each [but not ideal since $\mathcal{I}$ and $\mathbb{I}$ change based on parameterization of surface]. Instead look at principle curvatures since they are invariant to varying parameterizations.

Want to compute k_1, k_2 to show this.

(1) $k_1 = k_2 = 0$ [$\mathbb{R}^2$ is umbilic and 0].

(2) if radius = 1 $k_1 = 0$ [curvature along the cylinder and $k_2 = 1$ [curvature around the circle cross-section of the cylinder].
(3) Cone angle is 45 degrees: $k_1 = 0$ [vertically along the outside of the cone], $k_2 =$
Recall, we computed the curvatures for a revolutionary surface: $k_1 = \frac{f''(u)}{(1+f'(u)^2)^{3/2}} = 0[for cone]$ and $k_2 = \frac{1}{f(u)(1+f'(u)^2)^{1/2}} = \frac{1}{z(1+1)^{1/2}}$
So the curvatures are clearly different [independent of parameterization] $\Rightarrow$ not globally isometric.

Ex. Tangent Developable [surface using a spatial curve]
Show Gauss Curvautre of Tangent Developable $\kappa = 0$
Use the procedure we did today (adaptive frame): or score sum directly, **Try this out yourself**.

12 November 10

Sections 6.1 and 6.2 of donaldson.

12.1 Geodesic in $\mathbb{R}^3$

Theorem 12.1 *The shortest path between two points is a straight line under the Euclidean Metric.*

Proof: Assume curve $\vec{x}$ is the shortest path $\vec{x}(t)$ for $t \in [a, b]$. A variation of $\vec{x}$: $\vec{x}_\epsilon(t) = \vec{x}(t) + \epsilon * \vec{y}(t)$ where $\vec{y}(t)$ is a vector field along $\vec{x}(t)$, and $\vec{y}(a) = \vec{y}(b) = 0$.

The shortest path implies $\vec{x}(t)$ is a stationary curve $\mathcal{L}_\epsilon = Length(\vec{x}_\epsilon(t)$. $0 = \frac{d}{d\epsilon}_{\epsilon=0}\mathcal{L}(\epsilon)$ where ϵ is a critical point of $\mathcal{L}$.

So $\int_a^b \vec{x} \cdot \vec{y} dt = 0$ Since $\vec{y}$ is arbitrary, it follows that $\vec{x}'' = 0$ everywehre $\Rightarrow$ $\vec{x}$ is a straight line.

12.2 Geodesic in Surface

Generalize Stationary Curve from Euclidean space to curved surfaces.
Geodesic is a curve which has stationary length.
This means for any variation $\vec{x}_\epsilon(t) = \vec{x}(t) + \epsilon\vec{y}(t)$ where $\vec{y}(t) \perp \vec{n}$ [stays in the tangent plane to the surface], then $\frac{d}{d\epsilon}|\epsilon = 0 length(\vec{x}_\epsilon(t)) = 0$ [condition for this curve to be a geodesic: change in the length of the curve with respect to ϵ at 0 is 0].

Theorem 12.2 *A geodesic $\vec{x}$ satisfies $\frac{d^2}{dt^2}\vec{x}(z(t))$ is normal to the surface [does not have any tangential component].*

Generalization of stationary curve from euclidean space to a surface: second derivative is not 0 everywhere but instead does not have any component in the normal direction.
Proof: Assume $\vec{x}(z(t)$ has arclength. Then $0 = \frac{d}{d\epsilon}|_{\epsilon=0}\text{length}(\vec{x}_\epsilon(t)) = \frac{d}{d\epsilon}|_{\epsilon=0}\int_a^b |\vec{x}_\epsilon(t)|$

Theorem 12.3 *If a curve satisfies $\vec{x}''$ is normal, then it has constant speed and is a geodesic.*

Proof: $\frac{d}{dt}|\vec{x}'(t)|^2 = 2\vec{x}''(t)\cdot\vec{x}'(t)$ [$\vec{x}'(t)$ is in the tangent space], so $2\vec{x}''(t)\cdot\vec{x}'(t) = 0$ so $|\vec{x}'(t)| = constant$.
Therefore, we will assume a geodesic has unit speed.

Ex.
On the sphere, geodesics are great circles.
Proof: Suppose that we have a geodesic $\vec{x}(z(t))$. Assume the sphere has radius 1. $\vec{x} = \vec{n}$ [point on the sphere is equal to the normal vector at that point]. $\Rightarrow \vec{x}'' = f\cdot\vec{x}$
$\vec{x}''\cdot\vec{x} = f\cdot\vec{x}\cdot\vec{x}$ [and $\vec{x}\cdot\vec{x} = 1$ since we assume sphere has radius 1].
So we have $\vec{x}''\cdot\vec{x} = (\vec{x}'\cdot\vec{x})' - \vec{x}\cdot\vec{x}' = f$. Geodesic so $\vec{x}' =$ constant (constant velocity for geodesic) $= v^2$. Also $\vec{x}\cdot\vec{x}' = 0$ because $\vec{x}'$ is tangential and $\vec{x}$ is normal.
Therefore, $-\vec{x}'\cdot\vec{x}' = -v^2 = f$ and we see $\vec{x}'' = -v^2\vec{x}$.
vx'' is in the same direction as $\vec{x}$ so $\vec{x}'\times\vec{x}$ is a constant vector: $(\vec{x}'\times\vec{x})' = \vec{x}'\times\vec{x}' + \vec{x}''\times\vec{x} = 0 + 0 = 0$, so $\vec{x}\times x$ is a constant vector k (derivative of it is 0, so must be a constant. Therefore, $\vec{x}$ is the great circle orthogonal to k.

Question: Is a geodesic necessarily a shortest path?
Geodesic must have stationary length, but does this imply the Geodesic has minimal length?
Answer: Not always. Two geodesics along a circle through the diameter of the circle: one of them has shortest length. The geodesic with shortest length is called a "minimizing geodesic".

A geodesic is always **locally** length minimizing. Fix one point along the geodesic. Then any point sufficiently close has locally minimum length.

Theorem 12.4 *Given any point p and $v \in T_p\vec{x}$, can find a unique geodesic $\vec{x}(z(t))$ so that $\vec{x}'(0) = v$, $\vec{x}(0) = p$ for some interval $[0,\tau]$ consequence of the existence of solutions to 2nd order ODEs [just setting IC for second order ode].*

Theorem 12.5 *$\vec{x}(z(t))$ is a geodesic, then the following equations are satisfied:*
$$\begin{cases} \frac{d}{dt}\theta_1(z'(t)) + w_{12}(z')\theta_2(z') = 0 \\ \frac{d}{dt}(\theta_2(z'(t))) - w_{12}(z')\theta_1(z') = 0 \\ (\theta_1(z'))^2 + (\theta_2(z'))^2 = constant \end{cases}$$

First two are the "geodesic" equations and the last one to be the "energy equation".

$\vec{x}''$ is normal, so moving frame $\{e_1, e_2\}$ is an adaptive frame (in the tangent space), so $\vec{x}'' \cdot e_1 = 0 = \vec{x}'' \cdot e_2$. Therefore, $\vec{x}'' \cdot e_1 \Rightarrow \vec{x}' \cdot e_1 - \vec{x}' - e_1' = 0$
And $\vec{x}' = d\vec{x}(z') = (\theta_1 e_1 + \theta_2 e_2)(z') = \theta_1(z')e_1 + \theta_2(z')e_2$.
So $\vec{x} \cdot e_1 = \theta_1(z')'$.

Note The first two equations being satisfied are if and only if $\vec{x}$ is a geodesic. Also the first two equations imply the third. Geodesic equations $\Rightarrow$ energy equations.
$(\theta_1(z'))^2 + (\theta_2(z'))^2 = \mathcal{I}(z'(t), z'(t) = |\frac{d}{dt}\vec{x}(z(t))|^2$.
$\mathcal{I} = d\vec{x} \cdot d\vec{x} = (\theta_1 e_1 + \theta_2 \epsilon_2)(\theta_1 e_1 + \theta_2 e_2)$
And in an adaptive frame $I = t_1^2 + t_2^2$ so that $I(z', z') = \theta_1(z')^2 + \theta_2(z')^2 \iff \vec{x}(z(t))$ has constant speed.

Note: $d\vec{x}(z') = \frac{d}{dt}\vec{x}(z(t))$ [decomposition of surface map].

12.3 Geodesic Equation

13 November 15

13.1 Geodesics

Generalizing the straight line in Euclidean space to curved surfaces.

Def: Derivative of the curve projected on normal direction to the surface is 0: $\pi^{\perp}(D_\gamma \cdot \gamma) = 0$.
Any geodesic must satisfy the geodesic equations. The first two equations (the geodesic equations) imply the third is a constant. We take that constant to be 1 (so the geodesic has an arclength parameterization).

13.2 Hyperbolic Space

Upper-half plane model: $\mathbb{R} \times (0, \infty)$ with the metric given by $\mathcal{I} = \frac{dx^2 + dy^2}{y^2}$. By the Gauss theorem, the first fundamental form defines the Gauss curature (and everything else that's intrinsic about the curve).

Let $\theta_1 = \frac{dx}{y}$, $\theta_2 = \frac{dy}{y}$ so that we have forms $\mathcal{I} = \theta_1^2 + \theta_2^2$. $w_{12} =?$

Then using the structure equations $d\theta_1 + w_{12} \wedge \theta_2 = 0$ and $d\theta_2 - w_{12} \wedge \theta_1 = 0$, we have $d(\frac{dx}{y} + w_{12} \wedge \frac{dy}{y} = 0$ and $d\frac{dy}{y} - w_{12} \wedge \frac{dx}{y} = 0$.
The first euqation tells us $w_1 2$ is a multiple of dx, and we can plug this expression into the second equation to determine the unknown coefficient. After a bit of algebra, $w_{12} = -\frac{1}{y}dx$, and Gauss Curvature is $w_{12} = \kappa\theta_1 \wedge \theta_2 \Rightarrow \kappa = -1$

[has constant curvature of -1 everywhere (or negative constant curvature everywhere) – opposite of the standard sphere which has positive constant curvature].

Later on, we will determine the $\mathcal{I}$ as a "metric".
Consider geodesics on this space. Use the three equations to find them.

Assume $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a geodesic with arclength parameterization. Then it must satisfy the 2 geodesic equations and the energy equation. Can plug in $\theta_1, \theta_2, andw_{12}$ into geodesic equations to solve:

(1) $\frac{x'}{y}' - \frac{x'y'}{y^2} = 0$
(2) $\frac{y'}{y} + \frac{x'^2}{y^2} = 0$
(3) $\frac{x'^2}{y^2} + \frac{y'^2}{y^2} = 1$ (energy eqn)
$\theta_1 = \frac{dx}{y}$. So $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, so $z'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$ wiht $dx(\frac{\partial}{\partial x}) = 1$ and $dx(\frac{\partial}{\partial x}) = 0$. So $\theta_1(z') = \frac{dx}{y}(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}) = \frac{x'}{y}$

Now we have 3 equations (ODE) that we can solve.
(1) $\Rightarrow (\frac{x'}{y^2})' = 0$. This follows from $\frac{x'}{y^2}' = \frac{x''y^2 - x'2yy'}{(y^2)^2} = \frac{x''}{y^2} - \frac{x'2yy'}{y^4}$ ... $\frac{x''}{y} - \frac{2x'y'}{y^2} = 0$. Therefore $\frac{x'}{y^2} = A$ (i.e. a constant).
Case 1: $A = 0 \Rightarrow x' = 0 \Rightarrow x = C_2$. THerefore (3) $\Rightarrow \frac{y'^2}{y^2} = 1 \Rightarrow \frac{y'}{y} = \pm 1$, and therefore $y = C_1 \cdot e^{\pm t}$. Where C_1 is an arbitrary constant. $x(t) = C_2$ and $y(t) = C_1 e^{\pm t}$.

Consider $t = 0$, then $x(0) = C_2$ and $y(0) = C_1$, so this is the initial (starting point) of the geodesic and the $\pm t$ means it goes in either the positive or negative direction (two ways to traverse the line). So all vertical lines are geodesics.

Case 2: $A \neq 0$. (3) $\frac{y'^2}{x'^2} \Rightarrow 1 + \frac{y'^2}{x'^2} = \frac{y^2}{x'^2} = \frac{1}{A^2y^2}$ [replacing $x' = Ay^2$] and using $\frac{x'}{y^2} = A$. So we see that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$ [allows us to get rid of variable t to view it as a function of y and x]. Thus, $1 + (\frac{dy}{dx})^2 = \frac{1}{A^2y^2}$.

$\frac{dy}{dx} = \pm\sqrt{\frac{1-A^2y^2}{A^2y^2}}$ and $\frac{dx}{dy} = (\frac{dy}{dx})^{-1} = \pm\frac{Ay}{\sqrt{1-A^2y^2}}$, so $dx = \pm\frac{Ay}{\sqrt{1-A^2y^2}}dy$. Then we can integrate both sides to find
$x - C = \pm A^{-1}\sqrt{1 - A^2y^2}$.

$(x-c)^2 = A^{-2}(1 - A^2y^2)$ and $(x-c)^2 + y^2 = A^{-2} \neq 0$. which is a circle centered at $(C, 0)$ with radius $r = A^{-1}$, so the geodesic is a half-circle in our upper-half plane. C and A are any constants, so this geodesic encodes any half-circle of any radius with center on the x-axis.

For any given direction on the tangent plane, can find a unique geodesic with this direction as its initial velocity [geodesic entirely determined by initial point and then initial velocity].

In the hyperbolic space, you can not only solve the geodesic equation in a small neighborhood, but you can also extend it to an infinite length.

13.3 Covariant Derivatives

In E^3, a curve $\gamma(t)$ parameterized by $t \in I$ and we have a vector field $V(t)$, we say that $V(t)$ is parallel by $\frac{d}{dt}V(t) = 0$ [derivative of the vector field is 0 $\Rightarrow$ all pointing in teh same direction $\Rightarrow$ all parallel to each other].

Def. We say a tangent vectorfield $V(t)$ along a curve $z(t)$ on a surface $\vec{x}$ is parallel if (π is a projection) $\pi^T(\frac{d}{dt}V(t)) = 0$.

Def. Define $D_{\frac{d}{dt}}V(t) = \pi^T(\frac{d}{dt}V(t)$ is called the covariant derivative. Project the derivative onto the tangent space, and if the projection is 0 (i.e. doesn't have any velocity in the tangent direction), we say that it is a parallel vector field. This is equivalent to saying the covariant derivative is 0. Take derivative of vector field, and then project into the tangent space of the surface (where $\vec{x}(z(t))$ is a curve on the surface).

$\dot{\gamma} = \gamma'(t) = \frac{d}{dt}$. ($\frac{d}{dt}$ is in the interval [a,b] and then taken to U by γ (for simplicity, we say they are equivalent): $\gamma * \frac{d}{dt} = \gamma'(t)$ (modulus the change in magnitude induced by γ if γ is not arclength).

Theorem 13.1 *A curve γ in a surface $\vec{x}$ is a geodesic if and only if $D_\gamma \cdot \dot{\gamma} = 0$*

Proof: By definition $D_\gamma\dot{\gamma} = \pi^T(\dot{\dot{\gamma}}(t)) = 0 \iff$ geodesic.

Theorem 13.2 *Given a smooth curve γ on $\vec{x}$, and an initial vector v_0 tangent to the surface at $\gamma(t_0)$, there exists a unique parallel vector field so that $v(t_0) = v_0$.*

Proof: See donaldson. $v(t)$ is a vector field along $\gamma(t) = \vec{x}(z(t))$. e_1, e_2 : adaptive frame on the surface. $v(t)$ at every point is a tangent vector so it can be represented as a lin. comb. of e_1, e_2, $v(t) = a(t)e_1(z(t)) + b(t)e_2(z(t))$. $D_{\frac{d}{dt}}v = \pi^T\frac{d}{dt}v(t)$ ... $\pi^T(de_1) = w_{21}e_2$ [when projected to the tangent space, has only vector in the e_2 component].
...
$v(t) = a(t)e_1 + b(t)e_2$ gives the desired parallel vector field. We'll call it the parallel transport of $v(t_0)$ (i.e. $v(t_0)$ is the initial vector we specify).

Ex. Sphere example: of note is that when you make a cycle on the sphere, it's possible to end with a vector different from the one you started with: the parallel transport along a loop may change the initial vector [quite different from the Euclidean space – this will never happen]. In E^3, if you start with a vector and parallel transport it, you get the same vector you start with, but it's not the same story on the sphere (starting vector is different from ending vector after parallel transport).

Theorem 13.3 *The parallel transport along loops does not change the vector if and only if it has* $\kappa = 0$.

Somehow related to Gauss curvature: if Gauss curvature is 0, you can parallel transport across a loop without changing the ending vector from the starting one.

Theorem 13.4 *The parallel transport maintains the norm.*

If the vector field changes length (i.e. magnitude) it is not considered to be parallel. If you move it in the normal direction, then you leave the direction. We don't mind how the vector changes in the normal direction, as long as the vector has no tangent components, it is considered a parallel vector field.

14 November 17

Theorem 14.1 *The parallel transport does not change the initial vector if* $\kappa \equiv 0$

Proof: $\kappa = 0$. Therefore $dw_{12} = 0$ and w_{12} is a closed 1-form. $w_{12} = df$ for some function f. Set $v(t) = a(t)e_1(z(t)) + b(t)e_2(z(t))$ where $v(t)$ is the parallel transport of $v(0)$. $\begin{bmatrix} a'(t) \\ b'(t) \end{bmatrix} = \begin{bmatrix} 0 & -w_{12}(z') \\ w_{12}(z') & 0 \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$
Therefore $\begin{bmatrix} a(t) \\ b(t) \end{bmatrix} = \begin{bmatrix} \cos \int_z w_{12} & -\sin \int_z w_{12} \\ \sin \int_z w_{12} & \cos \int_z w_{12} \end{bmatrix}$ and via the fundamental theorem of calculus since we're integrating along a loop, the integral terms are 0: $\int_{\gamma_1} w_{12} - \int_{\gamma_2} w_{12} = \int_\Omega dw_{12} = 0$ where the two line integrals via Green's.

15 Clairaut's Theorem

Revolutionary surface: $\vec{x}(u, \phi) = \begin{bmatrix} f(u)\cos\phi \\ f(u)\sin\phi u \end{bmatrix}$.

$u \equiv$ constant (latitude meridian) is a geodesic if and only if $f_u = 0 \iff f'(u) = 0$ so that the geodesic equations are satisfied.

Equivalently, check if a curve satisfies $f\cos(\alpha)$ to determine if it is a geodesic.

15.1 Geodesic Curvature

Geodesic Curvature measures how far away a curve is from being a geodesic (so a geodesic has zero geodesic curvature).
Def. $\gamma(t)$ is a unit speed curve in $\vec{x} : U \to \mathbb{R}^3$. Define kg by: $D_{\frac{d}{dt}}\gamma' = kg \cdot \vec{n} \times \gamma'$. $D_{\frac{d}{dt}}\gamma' = \pi^T\gamma''$ (projection of the second derivative on the tangent space). Note $\gamma'' \perp \vec{n}, \gamma'$ because the curve is on the surface and orthogonal to γ' because γ' has unit speed. kg is the geodesic curvature is the norm of this cross product.

Theorem 15.1 $kg = \frac{(\gamma' \times \gamma'') \cdot \vec{n}}{|\gamma'|^3}$ *for any curve γ (not necessarily arclength).*

Proof:

Theorem 15.2 *Let $\gamma(t)$ a curve on the surface $\vec{x} : U \to \mathbb{R}^3$. Let $\vec{n} = \vec{n}(\gamma(0))$, $\vec{T} = \gamma'(0)$, then we have:*
1. Consider the curve $\hat{\gamma(t)} = \gamma(t) - < \gamma(t), \vec{n} > \cdot \vec{n}$ [projection of γ onto the tangent plane. – simply subtract away the normal component of the curve form it.]
$kg(0) =$ the planar curvature of $\hat{\gamma}(t) = |\pi^T\gamma'(0)|$
2. Consider $\hat{\gamma}(t) =< \gamma(t), \vec{T} > \cdot \vec{T} + < \gamma(t), \vec{n} > \vec{n}$ the projection onto the space spanned by $< T, \vec{n} >$. The normal curvature $k_n(0)$ is equal to the plane curvature of $\tilde{\gamma}(t)$ at $\tilde{\gamma}(0) = \gamma(0)$.
3. $\kappa^2 = kg^2 + k_n^2$ (curvature of the curve itself vs the curvature of the curve's projection onto two different planes). κ is the curvature of $\gamma(t)$ at $\gamma(0)$.

$\gamma''(t)$ not necessairly in the tangent plane – need to project it there. Planar curvature = projected curvature of second derivative for any curve?

16 November 29

16.1 Area Form

In multivariable calculus, define the area of sufrace $S \subseteq \mathbb{R}^3$, $\overline{X} : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$. And to compute the area, we use:

$$\int\int ||\vec{x}_u x \vec{x}_v|| dudv$$

Today: we would like to express this area in terms of the metric I on U. We have an induced metric on this surface onto this domain: $\mathcal{I} = Edu^2 + 2Fdudv + Gdv^2$ (want to only use the first fundamental form metric to express this area).

Suppose e_1, e_2 to be a positive orthogonal frame ($e_1 \times e_2 = \vec{n}$) for the tangent space T_qS. Then since e_1 and e_2 are normal to the surface..

- $\vec{x}_u = ae_1 + be_2$
- $\vec{x}_v = ce_1 + de_2$
- $$\begin{aligned}\vec{x}_u \times \vec{x}_v &= (ae_1 + be_2)x(ce_1 + de_2)\\ &= (ad - bc) \cdot \vec{n}\\ &= det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \vec{n}\end{aligned}$$

so $||\vec{x}_u \times \vec{x}_v|| = ||det \begin{bmatrix} a & b \\ c & d \end{bmatrix}|| = det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if we assume $\vec{x}$ is positively orientated (since $\vec{n}$ is a unit vector).

$\theta_i = e_i \cdot d\vec{x}$
$\theta_i(\frac{\partial}{\partial u}) = e_i \cdot d\vec{x}(\frac{\partial}{\partial u}) = e_i \cdot vx_u$.
$\theta_1(\frac{\partial}{\partial u}) = e_1 \cdot \vec{x}_u = a$
$\theta_2(\frac{\partial}{\partial u}) = e_2 \cdot \vec{x}_u = b$
$\theta_1(\frac{\partial}{\partial v}) = e_1 \cdot \vec{x}_v = c$
$\theta_2(\frac{\partial}{\partial v}) = e_2 \cdot \vec{x}_v = d$
$det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = det \begin{bmatrix} \theta_1(\frac{\partial}{\partial u}) & \theta_2(\frac{\partial}{\partial u}) \\ \theta_1(\frac{\partial}{\partial v}) & \theta_2(\frac{\partial}{\partial v}) \end{bmatrix}$
$= \theta_1 \wedge \theta_2(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$

Recall: k 1-forms, $\theta_1, ..., \theta_k$ and k vectors $u_1, ..., u_k$. With $\theta_1 \wedge ... \wedge \theta_k(u_1, ..., u_k) = det(\begin{bmatrix} \theta_1(u_1) & ... & \theta_k(u_1) \\ ... & & \\ \theta_1(u_k) & ... & \theta_k(u_k) \end{bmatrix}$

$\theta_1 \wedge \theta_2$ (two form on U) is called **area form** for metric $\mathcal{I}$ on U. $\theta_1 \wedge \theta_2(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = ||\vec{x}_u \times \vec{x}_v||$. $\Rightarrow \theta_1 \wedge \theta_2 = ||\vec{x}_u \times \vec{x}_v|| du \wedge dv$
Recall $du \wedge dv(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = 1$
and that $du \wedge dv$ is the basis of the space of 2-forms on U.
Two forms acting on the same vector (i.e. the basis vector) $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ measures the area difference (or change) from the 2-form $du \wedge dv$ to $\theta_1 \wedge \theta_2$.
So $\theta_1 \wedge \theta_2$ measures the (signed) area of rectangels in T_pU w.r.t. $\mathcal{I}$. Area(S) $= \int_U \theta_1 \wedge \theta_2 = \int ||\vec{x}_u \times \vec{x}_v|| dudv$. [clearly independent of the choice of paratemerization].

16.2 Minimal Surface

Consider the problem of finding a surface with smallest area (i.e. minimal surface) with a given boundary. For a curve in $\mathbb{R}^3$, which surface has boundary equal to the curve, but minimum area.

Recall how to find a geodesic. A Geodesic can be viewed as a 1-dimensional minimal surface. Fix two points, x, y and we'd like to find the minimal length curve that joins x, y. Same idea we'll use for minimal surfaces: give it a small variation, but the minimal one will be the cricital point.

Given $\vec{x} : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ with $\vec{x}(\partial U) = \alpha$. Where $\alpha(s)$ is a curve in $\mathbb{R}^3$.

Consider the normal variation: $\vec{x}_\epsilon(u, v) = \vec{x}(u, v) + \epsilon f(u, v) \cdot \vec{n}$: a small perturbation in the normal direction where f is a smooth function, $f = 0 on \partial U$. (∂U is the boundary of the domain). $\alpha = \partial S$.

We say that S has a stationary area if $\frac{d}{d\epsilon}|_{\epsilon=0}(area\vec{x}_\epsilon) = 0$ for all choices of f ($f|_{\partial U} = 0$). This is the 2d analogue of the Geodesic formulation (perturbations in a "locally" straight line form a curve).

What is $\mathcal{I}_\epsilon$ (first fundamental form of $\vec{x}_\epsilon$. $\mathcal{I}_\epsilon = d\vec{x}_\epsilon \cdot d\vec{x}_\epsilon = d\vec{x} \cdot d\vec{x} + df\epsilon(d\vec{x} \cdot d\vec{n}) +$ higher order terms (i.e. $\epsilon^2(d\vec{f} \cdot \vec{n})^2$ – throwing out this term). Where
$d\vec{x}_\epsilon = d\vec{x} + d(\epsilon \cdot f(u, v) \cdot \vec{n}). = d\vec{x} + \epsilon d(f \cdot \vec{n}) = d\vec{x} + \epsilon(df \cdot \vec{n} + f \cdot d\vec{n})$
$d\vec{x}_\epsilon = d\vec{x} + \epsilon(df \cdot \vec{n} + f \cdot dn)$ (note $d\vec{x} \cdot \vec{n} = 0$).

Observe $d\vec{x} \cdot d\vec{x} = \mathcal{I}_0$ and $\epsilon(d\vec{x} \cdot d\vec{n}) = -\mathbb{I}$. So $\mathcal{I}_\epsilon = \mathcal{I} - 2\epsilon f \cdot \mathbb{I}$ where $\mathcal{I}$ and $\mathbb{I}$ are the 1st and 2nd fundamental forms of $\vec{x}$. We have that $\mathcal{I} = \theta_1^2 + \theta_2^2$, $\mathbb{I} = -(w_{13} \cdot \theta_1 + w_{23} \cdot \theta_2)$. So $\mathcal{I}_\epsilon = \theta_1^2 + \theta_2^2 + 2\epsilon f(w_{13} \cdot \theta_1 + w_{23} \cdot \theta_2)$. So So $\mathcal{I}_\epsilon \approx (\theta_1 + \epsilon f w_{13})^2 + (\theta_2 + \epsilon f w_{23})^2 + O(\epsilon^2)$ where $O(\epsilon^2)$ represents forms with order at least ϵ^2 (i.e. higher order terms).
$\mathcal{I}_\epsilon = (\theta_1 + \epsilon f w_{13})^2 + (\theta_2 + \epsilon f w_{23})^2$. Let $\theta_{1,\epsilon} = (\theta_1 + \epsilon f w_{13})$ and $\theta_{2,\epsilon} = (\theta_2 + \epsilon f w_{23})$ so we have $\mathcal{I}_\epsilon = \theta_{1,\epsilon}^2 + \theta_{2,\epsilon}^2$. As the 1st fundamental form has this formulation, we can express $d\vec{x}_\epsilon = \theta_{1,\epsilon} e_{1,\epsilon} + \theta_{2,\epsilon} e_{2,\epsilon}$

And the area form for $\vec{x}_\epsilon$: $\theta_{1,\epsilon} \wedge \theta_{2,\epsilon} = (\theta_1 + \epsilon f w_{13}) \wedge (\theta_2 + \epsilon f w_{23}) = \theta_1 \wedge \theta_2 + \epsilon f(w_{13} \wedge \theta_2 + \theta_1 \wedge w_{23})$.

Write $w_{13} = a\theta_1 + b\theta_2$ and $w_{23} = b\theta_1 + c\theta_2$, and can get $\theta_{1,\epsilon} \wedge \theta_{2,\epsilon} = \theta_1 \wedge \theta_2 + f\epsilon(a + c)\theta_1 \wedge \theta_2$. Recall $H = -\frac{1}{2} tr(\begin{bmatrix} a & b \\ b & c \end{bmatrix}) = \frac{-1}{2}(a + c)$.

Then $\theta_{1,\epsilon} \wedge \theta_{2,\epsilon} = \theta_1 \wedge \theta_2 - 2f\epsilon H \theta_1 \wedge \theta_2 + O(\epsilon^2)$.

$\frac{d}{d\epsilon}|_{\epsilon=0}\theta_{1,\epsilon} \wedge \theta_{2,\epsilon} = 0 = -2fH\theta_1 \wedge \theta_2 + 0 \iff 2fH = 0$ true for all choices of $f \iff H = 0$ everywhere. This is the minimal surface equation [if a surface has stationary length].(may not always minimize the area – geodesic is a stationary length curve that does not always minimize the distance between two points on a surface).

Example: Catenoid $\begin{bmatrix} \cosh u \cos v \\ \cosh u \sin v \\ u \end{bmatrix}$.

$H = 0$ everywhere on the catenoid. Has mixed principle curvatures: κ_1 and κ_2 have different signs. More precise computation reveals $\kappa_1 + \kappa_2 = 0$. So this is a minimal surface.

Problem in our homework asking us to show the minimal surface of revolution are the catenoids (or variants of the catenoids). Only revolutionary surface that is a minimal surface. $H = 0$ is the only criterion of the minimal surface.

Want to define the integral of $n - forms$ w on an open set in $U \subseteq \mathbb{R}^n$. First, standard coordinates $x_1, ..., x_n \in \mathbb{R}^n$ (euclidean coordinates). $dx_1 \wedge .. \wedge dx_n$ is an n-form. We say any positive multiple of this to be positive $n - forms$.

Say that we choose a different coordinate in $\mathbb{R}^n$ $y_1, ..., y_n$ is a positive coordinate if $dy_1 \wedge .. \wedge dy_n$ is a positive n-form.

Now if we write this $y_i = y_i(x_1, ..., x_n)$, then $dy_1 \wedge ... \wedge dy_n = \det \left[\frac{\partial y_i}{\partial x_j} \right] dx_1 \wedge ... \wedge dx_n$.

Recall $\alpha_1 \wedge ... \wedge \alpha_k(v_1, ..., v_k) = \det \begin{bmatrix} \alpha_1(v_1) & ... & \alpha_k(v_1) \\ ... & ... & .. \\ \alpha_1(v_k) & ... & \alpha_k(v_k) \end{bmatrix}$

Any $n - form$ w in $\mathbb{R}^n$ can be written $w = h(x_1, .., x_n) dx_1 \wedge .. \wedge dx_n$ where $h(x_1, ..., x_n)$ is a function.
Define $\int_U w = \int_U h(x_1, ..., x_n) dx_1 \wedge ... \wedge dx_n$
by definition $= \int_U h(x_1, ..., x_n) dx_1 ... dx_n$
Note that this is well-defined, as it is independent of the choice of coordinates.

Suppose $w = g(y_1, .., y_n) \cdot dy_1 \wedge ... \wedge dy_n$. Then $\int_U w = \int_U g(y_1, ..., y_n) dy_1 ... dy_n$ [this is by the definition]. Goal is to show $\int_U w h(x_1, ..., x_n) dx_1 \wedge ... \wedge dx_n = \int_U h(x_1, ..., x_n) dx_1 ... dx_n$ because then we don't need to specifiy coordinates for the integral to hold (it is well-defined and independent of the choice of coordinates).

$dy_1 \wedge ... dy_n = det(\frac{\partial y_i}{\partial x_j} dx_1 \wedge ... \wedge dx_n$.
$w = g(y_1, ..., y_n) det(\frac{\partial y_i}{\partial x_j} dx_1 \wedge ... \wedge dx_n$
$= g(y_1(x_1, ..., x_n), y_n(x_1, ... x_n)) det(\frac{\partial y_i}{\partial x_j} dx_1 \wedge .. \wedge dx_n$
$= \int_U g(y_1, ..., y_n) dy_1 ... dy_n$ adn so on and so forth....

17 December 1

Last time: Integarl of $n-form$ w over $U \subseteq \mathbb{R}^n$ (n is same as degree of n-form w).
$w = h(x_1, ..., x_n)dx_1 \wedge .. \wedge dx_n$
Then define $\int_U w = \int_U h(x_1, ..., x_n)dx_1...dx_n$ if $x1...x_n$ is an orientated coordinate where ($dx_1 \wedge ... \wedge dx_n$ is positive). This is the definition.

Now we want to define the integral of 2-form on orientated surface $S \subseteq \mathbb{R}^3$. **The difference is this is not an open subset of** $\mathbb{R}^n$. It's an orientated surface.

Recall how we define integral of 1-form w over orientated curve α in $\mathbb{R}^n$.

$\int_\alpha w =?$ when w is a 1-form defined in $\mathbb{R}^3$. The integral of a 1-form along a curve: $\int_\alpha w = \int_a^b w(a'(t))dt$ where $\alpha'(t)$ is the tangent vector of α. Reparameterize by $\alpha(t(s))$, $s \in [c, d]$. Then $\int_\alpha w = \int_c^d w(\alpha'(t(s))t'(s))t'(s)ds = \int_a^b w(\alpha'(t))dt$ which means the integral is independent of parameterization.

Similarly, if we have w 2-form on region in $\mathbb{R}^3$ which contains the surface s (w is defined at least on an open neighborhood of this surface), can evaluate w on e_1, e_2 (orientated orthonormal basis at each point $p \in S$). Then integrate this function over S.
$\int_S := \int_S w(e_1, e_2)\theta_1 \wedge \theta_2$ where $\theta_1 \wedge \theta_2$ is the area (two-)form. [This is a definition].

Take a parameterization $\vec{x} : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$, and "pull back" w to get $\vec{x}^* w$ 2-form on U and integrate $\vec{x}^* w$ on U.

Independent of parameterization $\vec{x}$ as long as $\vec{x}$ is oriented). $\vec{x}^* w$: pull-back of the 2-form. The integral of an open set on a 2-form is simply $\int_U \vec{x}^* w$ as defined last class. This is a second definition of the integral of a 2-form on a $\mathbb{R}^3$ surface.

Claim: $\int_S w = \int_U \vec{x}^* w$

Rest of class: define $\vec{x}^* w$ pull-back.

Discussion of pull-back of k-forms from a surface.

Set-up: pull-back via a map (need a mapping). $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ (dimensions are arbitrary). such that f is a smooth map. Choose $x_1...x_m \in \mathbb{R}^m$ and $y_1...y_n$. Assume w is a k-form defined on a region that contains $f(U)$. Define $f^* w$ as the pull-back of w along f.

Define: $f^*w(v_1,...,v_k) = w(df(v_1),...df(v_k))$ where $v_1...,v_k$ are vectors in T_pU $\forall p \in U$. Effectively, we're moving k vectors from the tangent space such that $df(v_i) \in T_{f(p)}f(U)$.

(1) Note that when $m = 1$, (w is a 1-form) $U = [a,b]$, $f = \alpha(t)$. $(f^*w)(\frac{\partial}{\partial t}) = w(df(\frac{\partial}{\partial t})) = w(\alpha'(t))$, so this is the same as we defined before.

(2) Suppose $w = dg$. $g : \mathbb{R}^n \to \mathbb{R}$, (d is a 1-form). What is $f^*(dg)(v) = dg(df(v)) = d(g \circ f)(v)$.

As a consequence, we have $f^*(dy_i) = dy_i \circ df = d(y_i \circ f) = df_i$. $f(x_1...x_m) = \begin{bmatrix} f_1(x_1...x_m) \\ ... \\ f_n(x_1...x_m) \end{bmatrix}$. df is a vector-valued 1-form, so $df(V)$ is a vector.

(3) Now look at $w = \sum_{i=1}^n a_i(y_1,...,y_n)dy_i$ [general 1-form], then

Theorem 17.1 $f^*w = \sum_{i=1}^n a_i(f_1,...,f_n)df_i = \sum_{j=1}^m \sum_{i=1}^n a_i(f_1,...,f_n)\frac{\partial f_i}{\partial x_j}dx_j$

Proof: $f^*w = f^*(\sum_{i=1}^n a_I(y_1,...,y_n)dy_i) = \sum_{i=1}^n a_i(y_1,...,y_n)f^*(dy_i)$
$= \sum_{i=1}^n a_i(y_1,...,y_n)df_i$
$= \sum_{i=1}^n a_i(y_1...y_n)\sum_{j=1}^m \frac{\partial f_i}{\partial x_j}dx_j$
$y_i = f_i(x_1...x_n)$ so we have
$= \sum_{i=1}^n a_i(f_1...f_n)\sum_{j=1}^m \frac{\partial f_i}{\partial x_j}dx_j$

$(x_1...x_m) \to_f (y_1...,y_m)$.

(4) f^* preserves the wedge product. Let $\alpha_1, \alpha_2 : 1-forms$. Define $f^*(\alpha_1 \wedge \alpha_2) = (f^*\alpha_1) \wedge (f^*\alpha_2)$.
Proof:
$f^*(\alpha_1 \wedge \alpha_2)(v_1,v_2) = \alpha_1 \wedge \alpha_2(df(v_1),df(v_2)) = \det \begin{bmatrix} \alpha_1(df(v_1) & \alpha_1(df(v_2)) \\ \alpha_2(df(v_1)) & \alpha_2(df(v_2)) \end{bmatrix} = \det \begin{bmatrix} (f^*\alpha_1)(v_1) & (f^*\alpha_1)(v_2) \\ (f^*\alpha_2)(v_1) & (f^*\alpha_2)(v_2) \end{bmatrix} = (f^*\alpha_1) \wedge (f^*\alpha_2)(v_1,v_2)$

(5) $f^*w =?$ when $w = a(y_1...y_n)dy_{i_1} \wedge ... \wedge dy_{i_k}$.
$\sum_{i_1<...<i_k} a_{i_1...i_k}dy_{i_1} \wedge ... \wedge dy_{i_k}$ a general k-form.
$f^*w = a(f_1,...,f_n)df_{i_1} \wedge ... \wedge df_{i_n}$.

Example. Catenoid $f(u,v) = \begin{bmatrix} \cosh u \cos v \\ \cosh u \sin v \\ u \end{bmatrix}$. $w = (x_1^2 + x_2^2)dx_1 \wedge dx_2$ where $x_1x_2x_3$ are the standard coordinates in $\mathbb{R}^3$
(1) $f^*dx_1 = df_1 = d(\cosh u \cos v) = \sinh u \cos v du - \cosh u \sin v dv$
(2) $f^*dx_2 = df_2 = d(\cosh u \sin v) = \sinh u \sin v du + \cosh u \cos v dv$
(3) $f^*(w) = f^*(x_1^2+x_2^2)dx_1 \wedge dx_2) = (f_1^2+f_2^2)df_1 \wedge df_2 = \cosh^2 u(\sinh u \cosh u du \wedge dv) = \sinh u \cosh^3 u du \wedge dv$

Important Theorem:

Theorem 17.2 (Stoke's Theorem) *M is a compact orientated $k-$dimensional sub-manifold of $\mathbb{R}^n$, ∂M (boundary of the manifold) is also orientated), suppose w is a $(k-1)-form$, and then $\int_{\partial M} w = \int_M dw$.*

Explanation: ∂M has k-1 dimensions (boundary has 1 less dimensions. dw is a k form, and M is a k-dimensional surface.

Ex: $M = S$, surface $\subseteq \mathbb{R}^3$, ∂S =curve α. $w : 1 - form$. $\int_S dw = \int_{\partial S} w$. Integrate a 2-form on a surface or a 1-form on the boundary of the surface (i.e. a curve).

Ex. $M = B \subseteq \mathbb{R}^3$ is a compact domain in $\mathbb{R}^3$. $\partial B = S$. $w : 2 - form$. $\int_B dw = \int_{\partial B} w = \int_S w$.

Need to choose an orientation on the boundary, and an orientation of the manifold so the two orientations are compatible. $e_1 \times e_2 = \vec{n}$ along the boundary of the surface (normal needs to point outwards the surface as you traverse the boundary) – left hand side rule. $e_1 \times e_2$ gives the orientation of the boundary which is equal to the orientation of S (i.e. $\vec{n}$).

18 December 6

18.1 Pull-back of k-forms

Last time: $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$, $\{y_i\}$ is a coordinate, w is a $k - form$ on a neighborhood of $f(U)$. Pull-back f^*w k-form on U defined by $f^*w(V_1, ..., V_k) = w(df(v_1), .., df(v_k))$. Pushing forward k vectors form the domain U (i.e. $f^*w \subseteq \mathbb{R}^m \to f \to \mathbb{R}^n$).

Properties:
(1) if $w = dg, g : \mathbb{R} \to \mathbb{R}$, then $f^*(dg) = d(g \circ f)$. If $w = a_i(y)dy_i$ then $f^*w = (a_i \circ f)df_i$ where a_i are functions in y: $a_i(y_1, ..., y_n)$ [coefficients on y].

(2) $w = \alpha_1 \wedge ... \wedge \alpha_k$, $\alpha_i : 1 - forms$. $f^*w = f^*\alpha_1 \wedge ... \wedge f^*\alpha_k$.

With these two properties, we know what the pullback of a general k-form is (if the pullback is linear).

Today, a final special property (that will help us prove Stoke's Thm for special cases) (3) $f^*(dw) = d(f^*w)$: The pull-back is commutative with the exterior derivative.

Proof:
First, assume $w = a dy_1 \wedge ... \wedge dy_k$ where $a = a(y_1, ..., y_n)$. Prove single case. By linearailty, the rest will follow. First compute external derivative:

$dw = da \wedge dy_{i,1} \wedge ... \wedge dy_{i,k}$.
$f^*dw = f^*(da \wedge dy_{i,1} \wedge ... \wedge dy_{i,k})$
$= f^*(da) \wedge f^*(dy_{i,1}) \wedge ... \wedge f^*dy_{i,k}$
$= d(a \circ f) \wedge df_{i,1} \wedge ... \wedge df_{i,k}$
$f^*w = (a \circ f)df_{i,1} \wedge ... \wedge df_{i,k}$
$d(f^*w) = d(a \circ f) \wedge df_{i,1} \wedge ... \wedge df_{i,k} = f^*dw$
Recall $d^2 = 0$.

Take exterior derivative and pull-back is equivalent to taking the pull-back (and then the exterior derivative).

Theorem 18.1 (Stoke's Theorem) *M is a compact, oriented k-dimension sub-manifold of $\mathbb{R}^n$ with boundary ∂M equipped with boundary orientation. If w is a $(k-1) - form$ on M, then [integrating a n-form on a n-dimensional sub-manifold] $\int_M dw = \int_{\partial M} w$ [integrating a k-1 form on a k-1 sub-manifold].*

Proof:[For special Case]
Prove Stoke's Theorem when $k = 2$ and $M = S$ (M is a surface). Because $d(f^*w) = f^*(dw)$ and $\int_S dw = \int_U f^*dw$. when $S = f(U)$, f is a smooth $1-1$ map, it suffices to prove Stoke's thm for $U \subseteq \mathbb{R}^2$. This is because $\int_U f^*dw = \int_M dw = \int_{\partial M} w = \int_{\partial U} f^*w$ and from property (3), $\int_u d(f^*w)$. Pull back k-form to nicer subset in $\mathbb{R}^2 : U$ rather than a surface in $\mathbb{R}^3$. Uses $\int_a^b F(f(u))df(u) = \int_{f(a)}^{f(b)} F(f)df$.

(1) Suppose U is a topological disk [after deformations, you can deform this subset to a standard disk]. Proving for a single topological disk implies the theorem holds for topological disk with finitely many holes. Used 18.02 method for proving stoke's theorem (reduction by opposite orientated lines splitting up the surface).

(2) Suppose U is a region between two graphs $g_1(x) \leq y \leq g_2(x)$ where $a \leq x \leq b$. All topological disks can be divided into the regions in (2) [follows from implicit function theorem]. Therefore, only need to prove Stoke's for these "nicer" regions.

Goal: $\int_U dw = \int_{\partial U} w$. Assume $w = P(x,y)dx + Q(x,y)dy$. $dw = dP(x,y) \wedge dx + dQ(x,y) \wedge dy$
$dw = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$. Now $-\int_U \frac{\partial P}{\partial y} dx \wedge dy$ [assumed it was bounded by two graphs, so now can use multi-integration formula].
$= -\int_U \frac{\partial P}{\partial y} dxdy = -\int_a^b (\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)dy)dx = -\int_a^b P(x,y)|_{g_1(x)}^{g_2(x)} dx = -\int_a^b P(x, g_2(x) - P(x, g_1(x))dx$.
$\int_{\gamma_1} Pdx = \int_a^b P(x, g_1(x))dx$ [γ_1 is the curve along the boundary].
$\int_{\gamma_2} Pdx = -\int_a^b P(x, g_2(x))dx$ [γ_2 is the curve along the boundary].

So $-\int_a^b P(x,g_2(x)) - P(x,g_1(x))dx = \int_{\gamma_1\cup\gamma_2} Pdx = \int_{\partial U} Pdx$.

Now need to check to see if it's true for Q as well. Can simply prove $\int_U \frac{\partial Q}{\partial x} dx \wedge dy = \int_{\partial U} Qdy$.

Proved Stoke's Thm for $k = 2$ and $M = S$ (surface). This isn't exactly rigorous... For surfaces which are not topological disks.

Stoke's for higher dimensions: $k = 3$, $k-1 = 2$. 2-form in $\mathbb{R}^3$: w $=$ $Pdy\wedge dz + Qdz\wedge dx + Rdx\wedge dy$. Where $P(x,y,z)$. Can express any 2-form in this way as dx, dy, dz forms basis of 2-forms. Then $dw = (\frac{\partial P}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial P}{\partial x}) dx \wedge dy \wedge dz$ [3-form].

Consider compact domain $B \subseteq \mathbb{R}^3$. $\partial B = S$ is a smooth surface. Stoke's thm $\Rightarrow \int_B dw = \int_S w$.

$\int_B dw = \int_B (\frac{\partial P}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial P}{\partial x}) dx \wedge dy \wedge dz$. $\int_S w = \int_S Pdy\wedge dz + Qdz \wedge dx + Rdx \wedge dy = \int_S (P,Q,R)\cdot \vec{n} dA$.

Note $V = (P,Q,R)$, we say $divV = (\frac{\partial P}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial P}{\partial x})$ [called the divergence of a vector field V]. $\int_B divV = \int_S V \cdot \vec{n}$ [Special case of Stoke's is called the divergence theorem (or Gauss' theorem)].

Ex. Use Stoke's formula to compute the area of an ellipse: $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Recall: $\int_{\partial U} Pdx + Qdy = \int_U (-\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) dx \wedge dy$. Want to integrate some 2-form on this two dimensional space, but Green's tells we can convert this to a line integral on a 1-dimensional subspace. $x = 2\cos\theta$, $y = 3\sin\theta$ $\theta \in [0, 2\pi]$, so area in this domain is $A = \int_U dxdy = \int_U dx \wedge dy = \int_U d(xdy)$. Take $w = xdy$, then $\int_\alpha w = \int_0^{2\pi} 2\cos\theta 3\cos\theta d\theta = 6\pi$ So Area $= 6\pi$.

19 December 8

19.1 Stoke's

Stoke's formula. $w : (k-1)-$form in a neighborhood of M^k. $\int_M dw = \int_{\partial M} w$.

Ex. $\vec{x} : [0,\pi] \times [0, 2\pi] \to \mathbb{R}^3$. $\vec{x}(\phi,\theta) = \begin{bmatrix} a\sin\phi\cos\theta \\ b\sin\phi\sin\theta \\ c\cos\phi \end{bmatrix}$ where $\frac{x}{a}^2 + \frac{y}{b}^2 + \frac{z}{c}^2 = 1$.

Want to compute the area of the ellipsoid:

$takew = ddy\wedge dz$, then $dw = dx\wedge dy \wedge dz$. Then $\int_M dx\wedge dy\wedge dz = \int_{\partial M} xdy\wedge dz$ where LHS is the volume of M because ($dx \wedge dy \wedge dz$ is the Euclidean volume).

$= \int_M dxdydz$
(1) $xdy \wedge dz = ?d\theta \wedge d\phi$

(2) $\int_{\partial M} xdy \wedge dz = \int_0^{2\pi} \int_0^{\pi} ?d\theta \wedge d\phi$.
(3) Vol(M) $= \pi abc$, in particular, when $a = b = c = r$, then Vol(M) $= \pi r^3$
If this is a problem in the final exam, you'll need to use Stoke's formula.

Note the $w = xdy \wedge dz$ (or $-ydx \wedge dz$) These two two-forms are not equal to each other, but their integral on the curve are the same. If two two-forms have the same exterior derivative, then they must have the same integral by Stoke's formula.

Theorem 19.1 *If $dw = 0$, then $\int_{\partial M} = 0$.*

19.2 Gauss-Bonnet Theorem

Theorem in Riemann manifolds, it has higher-dimensional wordings, but in this class we focus on lower-dimensions.

Let S be an orientated, compact surface in $\mathbb{R}^3$ (with boundary).
Goal: Relate the integral of the Gauss Curvature: $\int_S kdA$ [dA is the area form] to some geometric data on the boundary as well as the topology of the surface.

Proposition 19.1.1 *$vx : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$. $\vec{x} = \vec{x}(u, v)$. Let $\alpha(s)$ be a curve in S [arclength parameterized].*

Curve on the definition domain: $\vec{x}(\gamma(s)) = \alpha(s)$ Curve from U mapped to S by $\vec{x}$.

Let e_1, e_2 be an adaptive frame on S. Then $\alpha'(s) = \cos(\phi(s))e_1 + \sin(\phi(s))e_2$. Can choose a function $\phi(s)$ to ensure the tangent vector has this form: $\alpha'(s) = a(s)e_1 + b(s)e_2$ and have $\alpha'(s) = 1 = a^2(s) + b^2(s) = 1$ since arclength parameterization. Can find $\phi(s)$ such that $\cos(\phi(s)) + \sin(\phi(s))$ equals 1. For instance, simply let $\phi(s) = arccos(a(s))$.

$\phi(s)$ is the angle between e_1 and $\alpha'(s)$.

Let w_{12} be the connection form determined by $\{e_1, e_2\}$ adaptive frame. Recall: $dw_{12} = k\theta_1 \wedge \theta_2$. We denote $\kappa_g(s)$ to be the geodesic curvature of $\alpha(s)$. Recall [curvature as a curve in $\mathbb{R}^3$ $\kappa^2 = \kappa_n^2 + \kappa_g^2$.

Proposition: $\kappa_g(s) = \phi'(s) - w_{12}(\gamma'(s))$.
w_{12} is a 1-form on domain U.
Proof:
$\alpha'(s) = T(s)$. Then $T'(s) = \kappa(s)N(s)$. $\alpha(s)$ is arclength, so this is the frenet eq. κ is the curvature of $\gamma(s)$. $\kappa(s)N(s) = \kappa_g(s)h(s) + \kappa_n(s)\vec{n}(s)$ has the previous

decomposition. $\vec{h}(s)$: is the unit vector tangent to S obtained by rotating $T(s)$ counter-clockwise by 90° [orthogonal to both T(s) and N(s)]. $\vec{n}$ is the surface normal. $\kappa_n = \kappa \cos\theta$ and $\kappa_g = \kappa \sin\theta$ is more precise than $\kappa^2 = \kappa_g^2 + \kappa_n^2$
$T'(s) = (-\sin(\phi(s))e_1 + \cos(\phi(s))e_2)\phi'(s) + \cos\phi(s)de_1(\alpha'(s)) + \sin\phi(s)de_2(\alpha'(s))$
[uses $T(s) = \alpha'(s) = \frac{d}{ds}\cos(\phi(s))e_1 + \sin(\phi(s))e_2$]

Where $de_1(\alpha'(s)) = \frac{d}{ds}e_1(\alpha(s))$

$T'(s) \cdot h(s) = \kappa(s)N(s) \cdot h(s) = \kappa_g(s)$
$k_g(s) = T'(s) \cdot \vec{h}(s) = (-\sin\phi(s)e_1 + \cos\phi(s)e_2)\phi'(s) \cdot h(s) + \cos(\phi(s))de_1(\alpha'(s)) + \sin\phi(s)de_2(\alpha'(s)) \cdot \vec{h}(s)$ Where $(-\sin\phi(s)e_1 + \cos\phi(s)e_2) = \vec{h}(s)$.
$T(s) : (\cos\phi(s), \sin(\phi(s))$
$\vec{h}(s) = (-\sin\phi(s), \cos\phi(s))$
$= \vec{h}(s) \cdot \phi(s) \cdot \vec{h}(s) + (\cos(\phi(s))de_1(\alpha'(s)) + \sin\phi(s)de_2(\alpha'(s))) \cdot h(s)$
$= \vec{h}(s) \cdot \phi(s) \cdot \vec{h}(s) + (\cos(\phi(s))de_1(\alpha'(s)) + \sin\phi(s)de_2(\alpha'(s))) \cdot (-\sin\phi(s)e_1 + \cos\phi(s)e_2$
and $de_1 \cdot e_2 = -w_{12} = -de_2 \cdot e_1$
$= \phi'(s) - w_{12}(\gamma'(s))$.

$d\vec{x}(\gamma'(s)) = \alpha'(s)$ where $\gamma'(s)$ is a pull-back of $\alpha'(s) \iff \alpha'(s)$ is push-forward of $\gamma'(s)$.

Proved that $\kappa_g(s) = \phi'(s) - w_{12}(\gamma'(s))$.

γ : closed curve. $\gamma : [a,b] \to U$ $\gamma(a) = \gamma(b)$. $\int_\gamma \kappa_g + \int_\gamma w_{12}(\gamma'(s)) = \int_\gamma \phi'(s) = 2\pi = \int_a^b \phi'(s)ds = \phi(b) - \phi(a)$. Here we choose e_1, e_2 to be the coordinate vector in $\mathbb{R}^2$: $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

Stoke's formula: $dw_{12} = k\theta_1 \wedge \theta_2$. So $\int_\gamma \kappa_g + \int_\omega \kappa\theta_1 \wedge \theta_2 = 2\pi$ where $\int_\omega \kappa\theta_1 \wedge \theta_2 = 0$ because Gauss curvature is 0 on the inside (b/c is a plane). Therefore, $\int_\gamma \kappa_g = 2\pi$ for a simple self-intersecting curve in $\mathbb{R}^2$. This is an invariant for curves (topological invariant).

More generally, if γ has intersection number $= k$, then $\int_\gamma \kappa_g = 2\kappa\pi$ (intersection number is the number of times it goes around s.t. both ends connect with each other).

Suppose you have another simple closed curve on the plane that is not smooth γ (is piece-wise smooth). Then $\int_\gamma \kappa_g + 0 = \int_a^b \phi'(s)ds \neq \phi(b) - \phi(a)$ because this function is not continuous. We actually have $\phi_1 + \phi_2 + \phi_3 + \int_a^b \phi'(s)ds = 2\pi$ [integral along the smooth part of the curve]. ϕ_1, ϕ_2, ϕ_3 is how much the angle changes on the non-smooth parts of the curve.

$\int_a^b \phi'(s)ds = 2\pi - (\phi_1 + \phi_2 + \phi_3) = \int_\gamma \kappa_g = \theta_1 + \theta_2 + \theta_3 - \pi$ $(\pi = \phi_i + \theta_i)$.

More generally, on a surface, if the curvature is κ, then $\int_\gamma \kappa_g + \int_\omega \kappa dA = \phi_1 + \phi_2 + \phi_3 - \pi$ [assume γ has 3 points of discontinuity where the surface bounded by the boundary is ω].

Gauss Bonnet Theorem: For any surface, $\int_S \kappa dA + \int_{\partial S} \kappa_g = 2\pi\chi(s)$. where $\chi(s)$ is called the Euler characteristic number of the surface. Euler characteristic number $= v - e + f$ v: number of vertices, e: number of sides, f: number of faces. $\chi(s) = 2 - 2\kappa$.

$$\chi(s) = \begin{cases} 1 \text{ disk} \\ 2 \text{ sphere} \\ 0 \text{ annulus} \end{cases}.$$

For surfaces without boundary, $\chi(s) = 2 - 2k$ [is given without boundary].